PHY905 Section 4
Ion Accelerator Technology
Lecture 7
- SRF Cavity Design-

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SRF Cavity Design

7.1 What is SRF cavity

7.2 Circuit model of RF cavity

7.3 Waveguide modes and Pillbox cavity

7.4 Characteristic parameters of SRF cavity

7.5 Criteria for cavity shape

7.6 Real cavity design using computer codes (single cell cavity)

7.7 High gradient cavity shape

7.8 HOM consideration for high current cavities

7.9 Multi-cell cavity design

7.10 Cavity frequency control in machine operation

7.11 Mechanical design

7.1 What is SRF cavity ?

- Principle of SRF cavity is not different from normal conducting RF cavities.
- RF heating is very small in SRF cavities, standing wave is used for beam acceleration.
- The cavity is designed for π-mode.

Principle of RF acceleration

TM-mode : Ez≠0, Bz=0, frequency: f

TM_{010} - mode, π-mode, Standing Wave

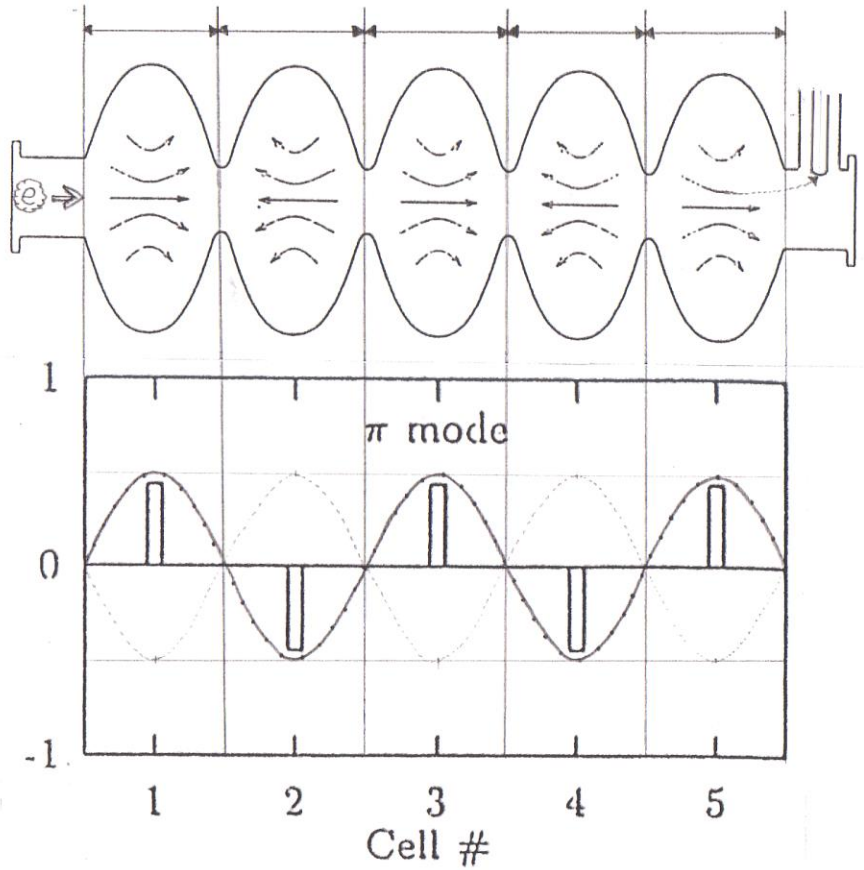
V(electron velocity)～C(light velocity)

L(cell length) = λ/2 ; λ(wave length)=C/f

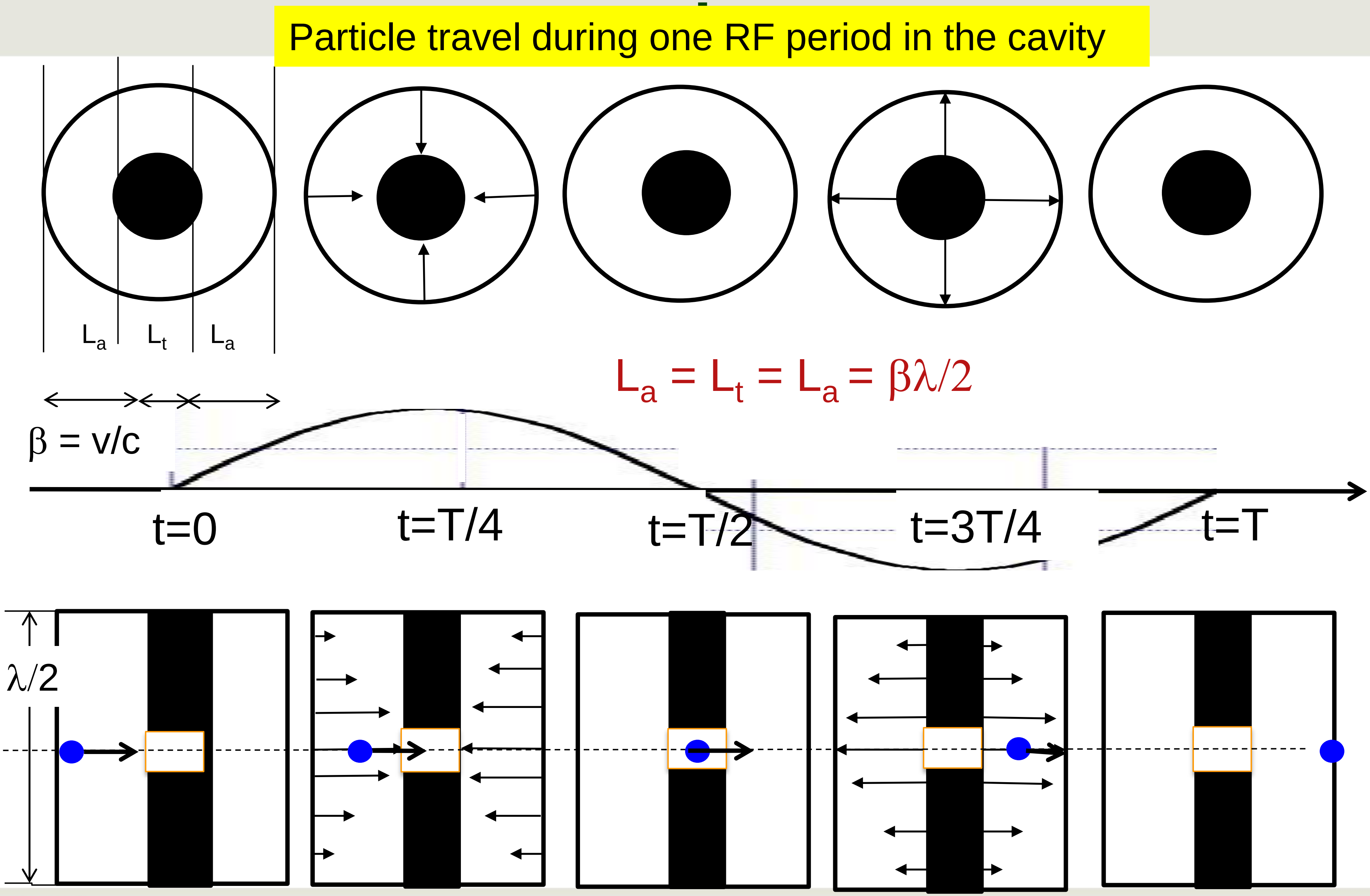
If the velocity is low like protons,

$\beta = V/C < 1$, then $L = \beta\lambda/2$

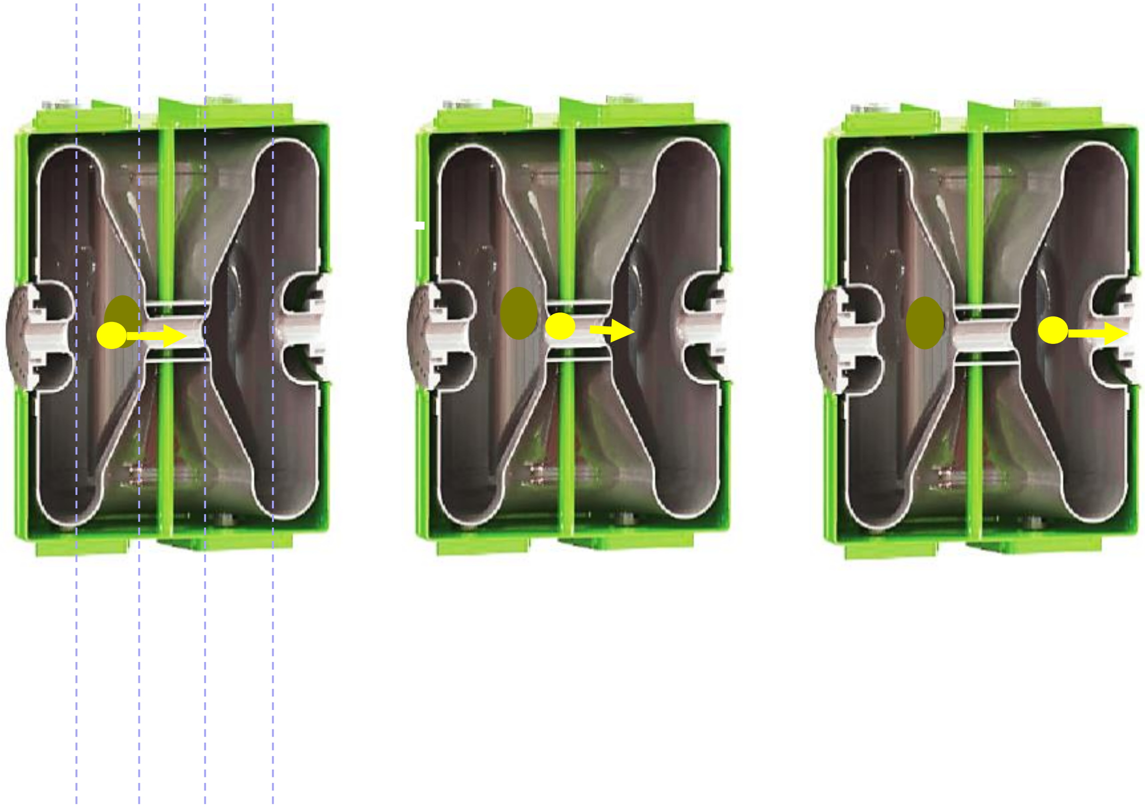
RF Cavity: accelerates charged particles by the electric field synchronized with RF frequency.



Low/medium beta cavity: FRIB half wave

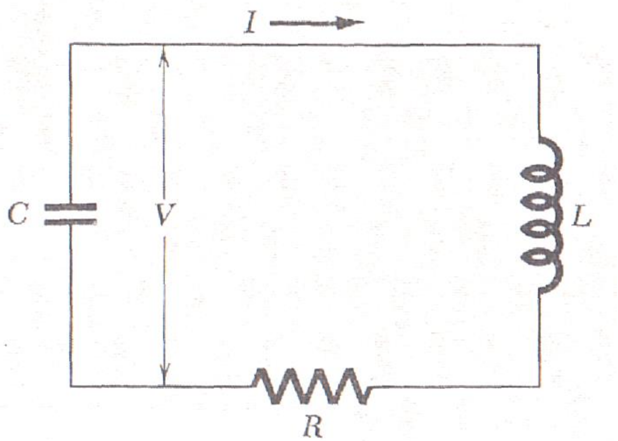


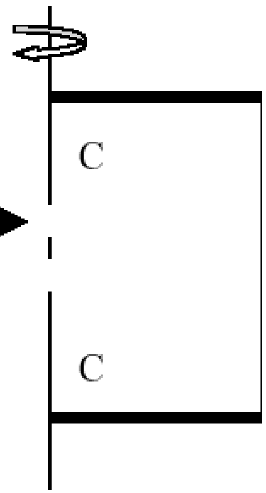
Heavy Ion Acceleration in HWR

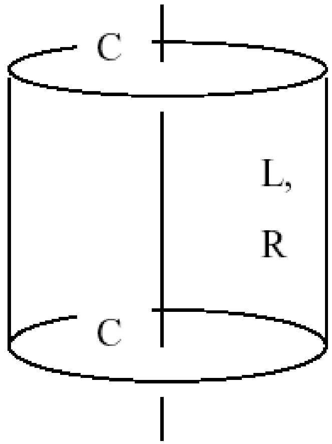


$L = \beta\lambda/2,\ t = T/2$

7.2 Equivalent circuit of the cavity







$$I = -\frac{dQ}{dt}, \quad Q = CV, \quad V = L\frac{dI}{dt} + RI$$

$$\frac{d^2V}{dt} + \left(\frac{R}{L}\right)\frac{dV}{dt} + \left(\frac{V}{LC}\right) = 0, \quad V(t) = V_O \exp(-\alpha + i\omega)t$$

$$(-\alpha + i\omega)^2 + (-\alpha + i\omega)\left(\frac{R}{L}\right) + \left(\frac{1}{LC}\right) = 0,$$

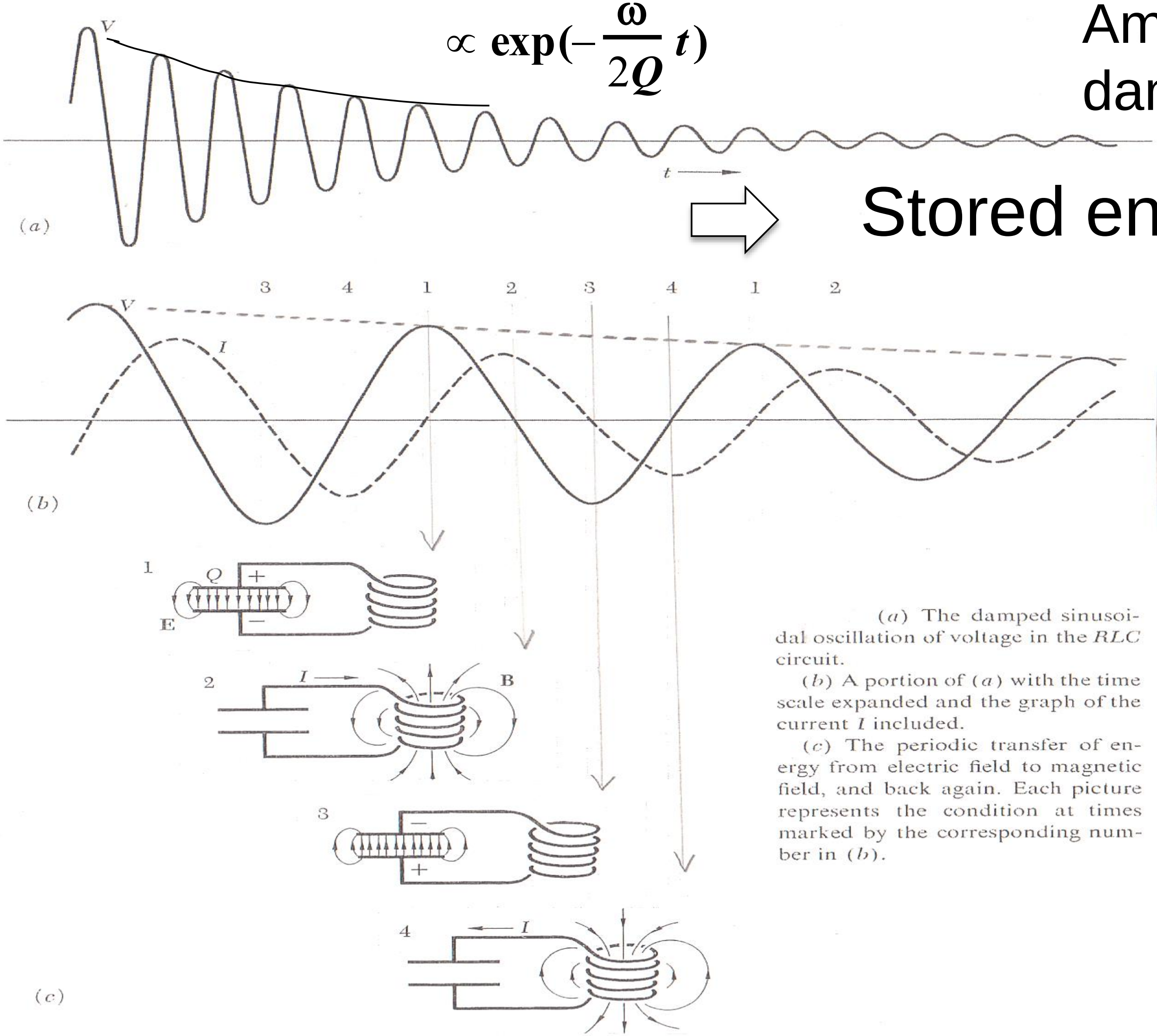
$$\alpha = \frac{R}{2L}, \quad \omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

$$R << L, \quad \omega_O^2 = \frac{1}{LC} \Rightarrow f = \frac{1}{2\pi\sqrt{LC}}$$

Simple circuit model of RF cavity
- Oscillation in the LCR circuit -

$$\propto \exp(-\frac{\omega}{2Q}t)$$

Amplitude damps by a damping constant 2Q.



(a) The damped sinusoidal oscillation of voltage in the RLC circuit.
(b) A portion of (a) with the time scale expanded and the graph of the current I included.
(c) The periodic transfer of energy from electric field to magnetic field, and back again. Each picture represents the condition at times marked by the corresponding number in (b).

Stored energy damps

$$\propto \exp(-\frac{\omega t}{Q})$$

Q: damping constant of the stored energy

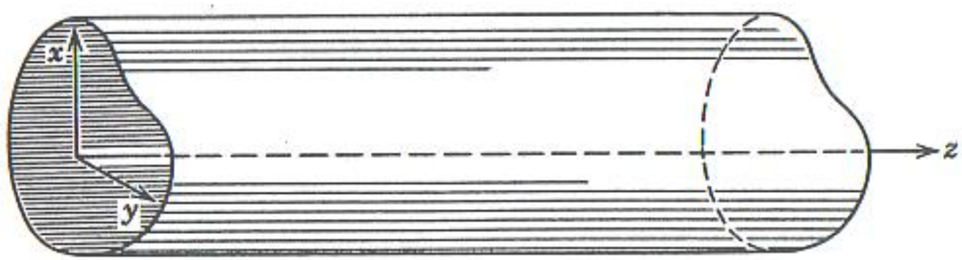
7.3 Waveguide modes and Pillbox cavity
- Mathematics formalism -

Maxwell equations in a waveguide

See Jackson's textbook

$$\vec{\nabla}\times\vec{E}=i\frac{\omega}{c}\vec{B},\quad \vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\times\vec{B}=-i\mu\varepsilon\frac{\omega}{c}\vec{E},\quad \vec{\nabla}\cdot\vec{E}=0,\quad \rho=0,\ j=0$$

$$\left(\vec{\nabla}^2+\mu\varepsilon\frac{\omega^2}{c^2}\right)\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix}=0,$$



$$\vec{E}(x,y,z,t)=\vec{E}(x,y)\exp(\pm ikz-i\omega t),\quad k\text{: wave vector}$$

$$\vec{B}(x,y,z,t)=\vec{B}(x,y)\exp(\pm ikz-i\omega t)$$

$$\left[\vec{\nabla}_t^2+(\varepsilon\mu\frac{\omega^2}{c^2}-k^2)\right]\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix}=0,\quad \vec{\nabla}_t^2\equiv\vec{\nabla}^2-\frac{\partial^2}{\partial^2 z},\quad \vec{E}=E_z\vec{e}_z+\vec{E}_t,\quad \vec{B}=B_z\vec{e}_z+\vec{B}_t$$

$$\vec{B}_t=\frac{1}{(\mu\varepsilon\frac{\omega^2}{c^2}-k^2)}\left[\vec{\nabla}_t\left(\frac{\partial B_z}{\partial z}\right)+i\mu\varepsilon\frac{\omega}{c}\vec{e}_z\times\vec{\nabla}_t E_z\right],$$

$$\vec{E}_t=\frac{1}{(\mu\varepsilon\frac{\omega^2}{c^2}-k^2)}\left[\vec{\nabla}_t\left(\frac{\partial E_z}{\partial z}\right)-i\frac{\omega}{c}\vec{e}_z\times\vec{\nabla}_t B_z\right]$$

TM(Transverse magnetic)/TE (Transverse electric) mode assignment of electro-magnetic field

TM-mode : $B_z = 0,\ E_z \neq 0$ → Can accelerate beam

$$\vec{B}_t = \frac{i\varepsilon\mu\frac{\omega}{c}}{(\varepsilon\mu\frac{\omega^2}{c^2} - k^2)} \vec{e}_z \times \vec{\nabla}_t E_z$$

$$\vec{E}_t = \frac{1}{(\varepsilon\mu\frac{\omega^2}{c^2} - k^2)} \vec{\nabla}_t \left(\frac{\partial E_z}{\partial z} \right)$$

→ Solving eigenfunction, fix k, Ez

$$\vec{\nabla}_t^2 E_z + \left(\varepsilon\mu\frac{\omega^2}{c^2} - k^2 \right) E_z = 0 ,$$

Boundary condition $E_z|_S = 0$ ($\because$ $\vec{n} \times \vec{E} = 0$ **on the surface of perfect conductor)**

$\frac{\partial B_z}{\partial n}|_S = 0$ ($n \cdot B = 0$ **on the surface,**

but automatically satisfied by the TM - mode condition)

TE-mode

TE-mode : $E_z = 0\ ,\ B_z \neq 0$ → Kicks beam to tranverse direction

$$\vec{B}_t = \frac{i\varepsilon\mu\frac{\omega}{c}}{(\varepsilon\mu\frac{\omega^2}{c^2} - k^2)}\vec{\nabla}_t\left(\frac{\partial B_z}{\partial z}\right)$$

$$\vec{E}_t = \frac{-i\frac{\omega}{c}}{(\varepsilon\mu\frac{\omega^2}{c^2} - k^2)}\vec{e}_z \times \vec{\nabla}_t B_z$$

$$\vec{\nabla}_t^2 B_z + \left(\varepsilon\mu\frac{\omega^2}{c^2} - k^2\right)B_z = 0\ ,$$

Boundary condition $E_z|_S = 0$ ($\vec{n} \times \vec{E} = 0$ **on the surface of perfect conductor but automatically satisfied by the TE- mode condition)**

$$\frac{\partial B_z}{\partial n}\Big|_S = 0\ (\because n \cdot B = 0 \text{ on the surface})$$

Solve eigenvalue problem

$$\psi(x,y)=E_z(x,y) \text{ for TM-mode or } B_z(x,y) \text{ for TE-mode}$$

$$\left(\vec{\nabla}_t^{\,2}+\gamma^2\right)\psi=0\,,\quad \psi|_S=0 \text{ (for TM-mode) or } \frac{\partial}{\partial n}\psi|_S=0 \text{ (for TE-mode)}$$

$$\gamma^2=\varepsilon\mu\frac{\omega^2}{c^2}-k^2\geq 0$$

From the boundary condition,

$$\gamma^2=\gamma_\lambda^2\,,\ \psi=\psi_\lambda\quad(\lambda=1,2,\cdots)\ \Rightarrow\ k_\lambda^2=\varepsilon\mu\frac{\omega^2}{c^2}-\gamma_\lambda^2$$

If $\omega < c\dfrac{\gamma_\lambda}{\sqrt{\varepsilon\mu}}$, then k_λ is an imaginal number. The wave is damped in the wavegu

$$\omega_\lambda=c\frac{\gamma_\lambda}{\sqrt{\varepsilon\mu}}\cdots\text{cutoff frequency}$$

When $\omega\geq\omega_\lambda$, wave number k_λ is a real number,

then the wave can propagate into the waveguide.

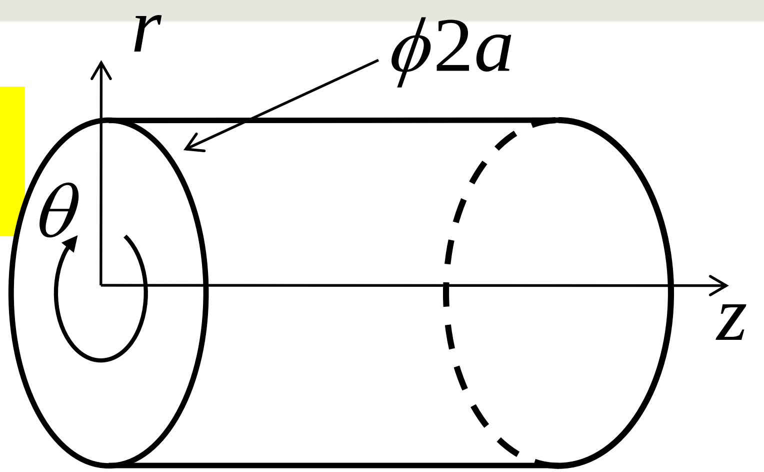
Application of the mathematics for circular waveguide

Plane wave propagating to z-direction

$$\begin{cases} \vec{E} = \vec{E}_0 e^{-i(kz-\omega t)} \\ \vec{H} = \vec{H}_0 e^{-i(kz-\omega t)} \end{cases}$$

k:imaginary ⇒ amplitude damps
k:real ⇒ propagates without damping

ω : angular frequency k : wave number



Decomposing longitudinal and transverse and put into Maxwell equation

$$\Delta \vec{E}_z + \gamma^2 \vec{E}_z = 0 \quad \Delta \vec{H}_z + \gamma^2 \vec{H}_z = 0 \quad \gamma^2 = \omega^2 \varepsilon_0 \mu_0 - k^2$$ (γ : propagation constant)

Eigen value problem of γ⇒ fixed γ, $k^2 = \omega^2 \varepsilon_0 \mu_0 - \gamma^2$ Get condition for wave propagation

$$\vec{E}_j = -\frac{1}{\gamma^2}\left(-i\omega\mu_0 \cdot \vec{z} \times \vec{\nabla} H_z + k \cdot \vec{\nabla} E_z\right)_j \qquad \vec{H}_j = -\frac{1}{\gamma^2}\left(k \cdot \vec{\nabla} H_z + i\omega\varepsilon_0 \cdot \vec{z} \times \vec{\nabla} E_z\right)_j$$

① Et, Ht both are decided by only Ez, Hz.
② Case of Hz=0,Ez≠0, Et and Ht can be written by Ez only.（TM mode, acceleration mode）
③ Case of Hz≠0,Ez=0, Et and Ht can be written by Hz only （TE mode, kick mode）

TM mode waves propagate in a circular waveguide

Cyclical coordinate

$$\nabla_t^2 E_z + \gamma^2 E_z = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \gamma^2 \right) E_z = 0$$

$E_z = R(r)\Theta(\theta)$ supposed $\Rightarrow$ $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \gamma^2 \right) R(r)\Theta(\theta) = 0$

$$\Rightarrow \frac{r^2}{R(r)}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \gamma^2 \right) R(r) = -\frac{1}{\Theta(\theta)}\frac{\partial^2}{\partial \theta^2}\Theta(\theta) \quad \left(= \nu^2 : \text{constant} \right)$$

Here, set ρ=γr, the equation is **Bessel equation**。

Bessel equation: $$\frac{d^2 R(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dR(\rho)}{d\rho} + \left(1 - \frac{\nu^2}{\rho^2} \right) R(\rho) = 0$$

$$\frac{d^2\Theta(\theta)}{d\theta^2} + \nu^2\Theta(\theta) = 0$$

Single harmonic motion

Function should be connected continuously at one frequency ⇒ m integer

General solution

$$R(r) \propto J_m(\gamma r), \quad \Theta(\theta) \propto e^{im\theta}$$

Early equation $$E_z = E_0 e^{im\theta} J_m(\gamma r) e^{-i(kz - \omega t)}$$

Boundary condition for circular waveguide

Perfect conductor surface、
$E_{\perp} \neq 0$,$E_{//} = 0$, $H_{\perp} = 0$, $H_{//} \neq 0$
$\Rightarrow$ γ fixs

$$at\ r = a, \quad J_m(\gamma a) = 0$$

$$\Rightarrow \gamma a = \rho_{mn} \Rightarrow \gamma = \frac{\rho_{mn}}{a}$$

$$\Rightarrow \gamma r = \frac{\rho_{mn}}{a} r$$

Electric field in parallel to waveguide surface should be zero.

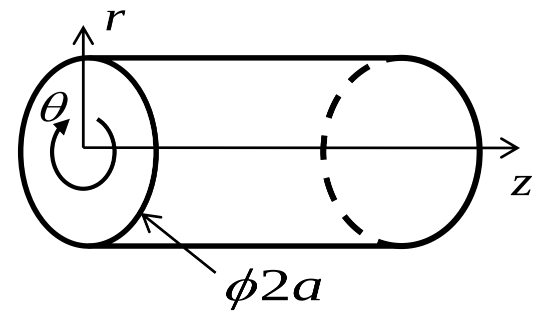
Ez in circular waveguide

$$E_z = E_0 e^{im\theta} J_m\left(\frac{\rho_{mn}}{a} r\right) e^{-ikz}$$

Bessel function of m-order

P_{mn} is nth root

m: number of node in Θ direction
n: number of node in r direction

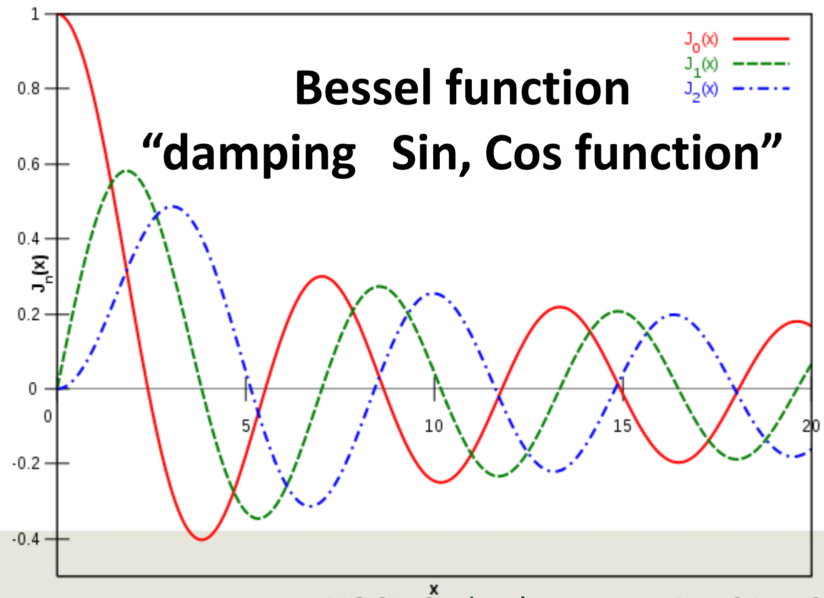


Put Ez into Maxwell equation to fix Er and Eθ

$$\vec{E}_t = -\frac{1}{\gamma^2}\left(-i\omega\mu_0 \cdot \nabla_t \times H_z + k \cdot \nabla_t E_z\right)$$

$$E_r = \frac{ik}{\gamma^2}\frac{\partial E_z}{\partial r} = -\frac{ik}{\gamma^2}\frac{\rho_{mn}}{a} E_0 e^{im\theta} J_m'\left(\frac{\rho_{mn}}{a} r\right) e^{-ikz}$$

$$E_\theta = \frac{ik}{\gamma^2}\frac{1}{r}\frac{\partial E_z}{\partial \theta} = \frac{ik}{\gamma^2}\frac{im}{r} E_0 e^{im\theta} J_m\left(\frac{\rho_{mn}}{a} r\right) e^{-ikz}$$



Cutoff frequency of TM mode function in a circular waveguide

if k not real : $\vec{E} = \vec{E}_0 e^{-i(kz-\omega t)}$ can not propagates (damps).

Eigenvalue γ is fixed by boundary condition $\gamma = \dfrac{\rho_{mn}}{a}$

⇨ Fix the condition of ω to propagates wave

$$\gamma^2 = \omega^2 \varepsilon_0 \mu_0 - k^2 \Rightarrow k^2 = \omega^2 \varepsilon_0 \mu_0 - \gamma^2 > 0 \Rightarrow \omega^2 > \frac{1}{\varepsilon_0 \mu_0} \gamma^2 = c^2 \left(\frac{\rho_{mn}}{a} \right)^2$$

➡ $\omega = \dfrac{c\rho_{mn}}{a}$ Called cutoff frequency, below this value wave can not propagate in the waveguide

Cutoff frequency : $f_{Cutoff} = \dfrac{c\rho_{mn}}{2\pi a}$

For instance circular waveguide with Φ60mm dia., Cutoff of TM_{mn}-mode below

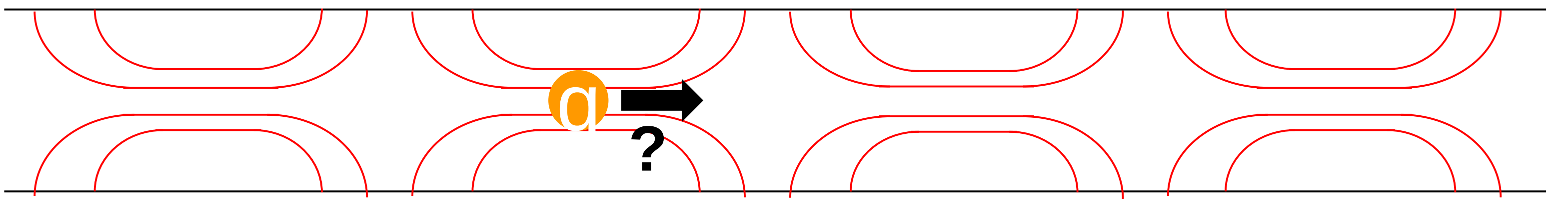
Mode	Cutoff Freqency[MHz]
TM01	3825
TM11	6095
TM21	8169
TM22	8779
TM21	11159
TM22	13387

$J_m(x)$ nth zero point $x=\rho_{mn}$

m \ n	1	2	3
0	2.405	5.520	8.654
1	3.832	7.016	10.173
2	5.136	8.417	11.620

Can accelerate charged particle by a Circular Waveguide ?

For instance, TM01 Mode is excited in a circular waveguide



TM01 function $E_z = E_0 J_0\left(\frac{\rho_{01}}{a} r\right) e^{-i(kz-\omega t)}$ k should be real number to propagate wave

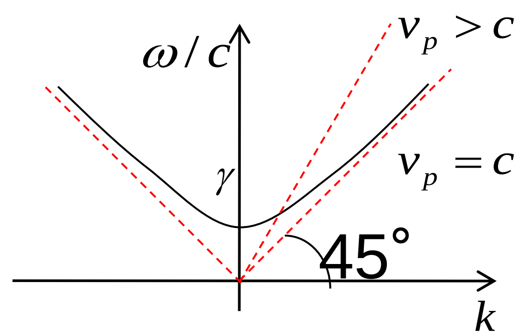
k and ω has a relationship fixed by the γ decided the waveguide shape

$$\gamma^2 = \omega^2 \varepsilon_0 \overset{v_p}{\mu_0} - k^2 = \left(\frac{\omega}{c}\right)^2 - k^2 \Rightarrow \begin{cases} k^2 = \left(\frac{\omega}{c}\right)^2 - \gamma^2 \\ \left(\frac{\omega}{c}\right)^2 = k^2 + \gamma^2 \end{cases}$$

If wave can propagate, γ is smaller than ω/c

at $kz = \omega t$, the field is maximum.

Phase velocity: $v_p = \frac{z}{t} = \frac{\omega}{k} = \frac{\omega}{\sqrt{(\omega/c)^2 - \gamma^2}} > c$, the maximum point runs at the velocity v_p.



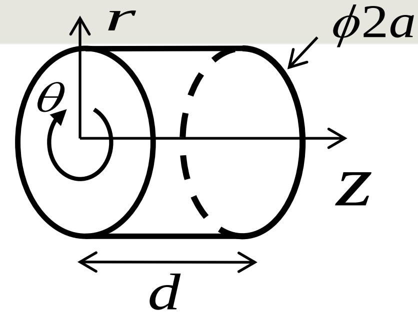
The phase of the electric field runs at the speed > c (speed of light)

⇨ Impossible to accelerate

Pillbox cavity design

Put a cover at both ends of circular waveguide ➡ additional boundary condition (z-direction) on the covers

At the covers : $z = 0$, at d $\quad E_t = 0 \quad E_r = E_\theta = 0$



Transverse electric field component R on the covers

$$E_r = -\frac{ik}{\gamma^2}\frac{\rho_{mn}}{a}E_0 e^{im\theta} J_m'\left(\frac{\rho_{mn}}{a}r\right)e^{-ikz} = -A\sin kz + iA\cos kz$$

Take the real part ➡ $\mathrm{Re}[E_r(z)] = -A\sin kz$

$$\mathrm{Re}[E_r(0)] = -A\sin 0 = 0$$
$$\mathrm{Re}[E_r(d)] = -A\sin kd$$

$$\sin kd = 0 \Rightarrow kd = p\pi \Rightarrow k = \frac{p\pi}{d} \quad \text{(p integer)}$$

Standing wave in z-direction, p is node number in z-direction

TM_{mnp} mode Take real part

$$E_r = -\frac{1}{\gamma^2}\frac{p\pi}{d}\frac{\rho_{mn}}{a}E_0 J_m'\left(\frac{\rho_{mn}}{a}r\right)\cos(m\theta)\sin\left(\frac{p\pi z}{d}\right) \qquad H_r = -\frac{\omega\varepsilon_0}{\gamma^2}\frac{m}{r}E_0 J_m\left(\frac{\rho_{mn}}{a}r\right)\sin(m\theta)\cos\left(\frac{p\pi z}{d}\right)$$

$$E_\theta = \frac{1}{\gamma^2}\frac{p\pi}{d}\frac{m}{r}E_0 J_m\left(\frac{\rho_{mn}}{a}r\right)\sin(m\theta)\sin\left(\frac{p\pi z}{d}\right) \qquad H_\theta = -\frac{\omega\varepsilon_0}{\gamma^2}\frac{j_{mn}}{a}E_0 J_m'\left(\frac{\rho_{mn}}{a}r\right)\cos(m\theta)\cos\left(\frac{p\pi z}{d}\right)$$

$$E_z = E_0 J_m\left(\frac{\rho_{mn}}{a}r\right)\cos(m\theta)\cos\left(\frac{p\pi z}{d}\right) \qquad H_z = 0$$

Resonant frequency $\quad \omega^2\varepsilon\mu = k^2 + \gamma^2 = \left(\frac{\rho_{mn}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2$

TM Mode frequency

$$\omega^2\varepsilon_0\mu_0 = \left(\frac{\rho_{mn}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2 \Rightarrow f = \frac{c}{2\pi}\sqrt{\left(\frac{\rho_{mn}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

d is arbitrary but consider the multi-cell cavity for particles run at speed of light.

Example of L-band

Fix the radius r and length d of the Pill box cavity to for the TM010-mode frequency to be 1300MHz

$$a = \frac{c\rho_{01}}{2\pi f} = 88.3[mm],\ d = \frac{c}{2f} = 115.3[mm]$$

$$c = 2.99792458\times10^8[m/s]$$

nth zero point of $J_m(x)$: $x=\rho_{mn}$

m \ n	1	2	3
0	2.405	5.520	8.654
1	3.832	7.016	10.173
2	5.136	8.417	11.620

Example

L-band cavity with beam pipe with 30mm radius at 1300MHz (TM010)

TM mnp	f [MHz]	Beam pipe
TM 0 1 0	1300	Reject
TM 0 1 1	1838	Reject
TM 1 1 0	2071	Reject
TM 1 1 1	2446	Reject
TM 2 1 0	2776	Reject
TM 0 1 2	2907	Reject
TM 0 2 0	2984	Reject

TM mnp	f [MHz]	Beampipe
TM 0 1 3	4111	Propagate
TM 1 1 5	6822	Propagate
TM 2 1 6	8279	Propagate

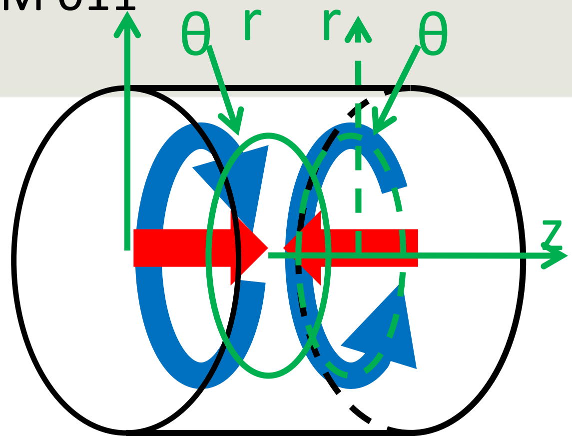
Lower frequency modes are trapped in the cavity, which is harmful to beam acceleration: Trapped mode

Cutoff of 30mm radius beam pipe

Mode	Cutoff [MHz]
TM01	3825
TM11	6095
TM21	8169

Filed distribution of TM_{mnp} modes example

TM 011



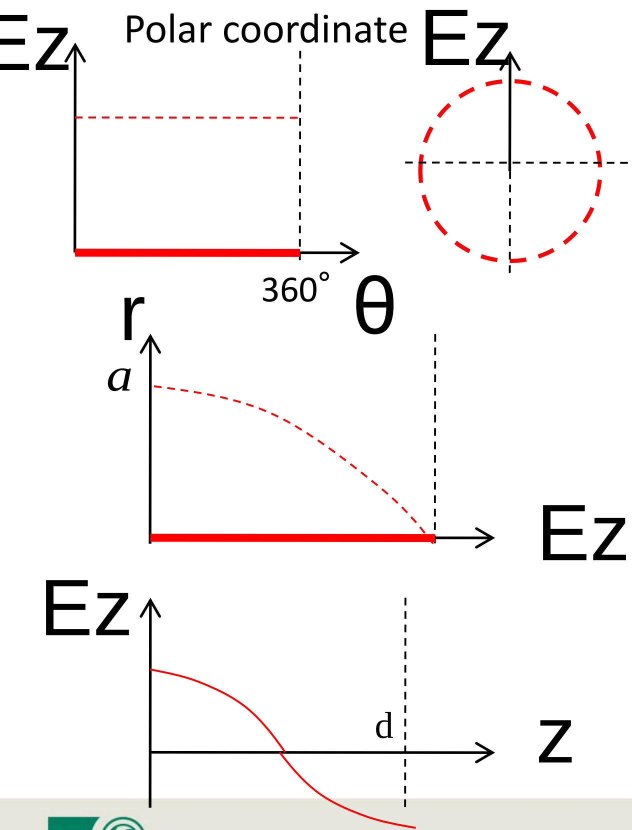
Draw the TM Mode field distribution in a Pillbox

$$E_z = E_0 J_m\left(\frac{\rho_{mn}}{a} r\right)\cos(m\theta)\cos(\frac{p\pi z}{d})$$

θ–direction : $\cos(m\theta)$

r-direction : $J_m\left(\frac{\rho_{mn}}{a} r\right)$

Z-direction : $\cos(\frac{p\pi z}{d})$



TM010

TM011

TM110

TM111

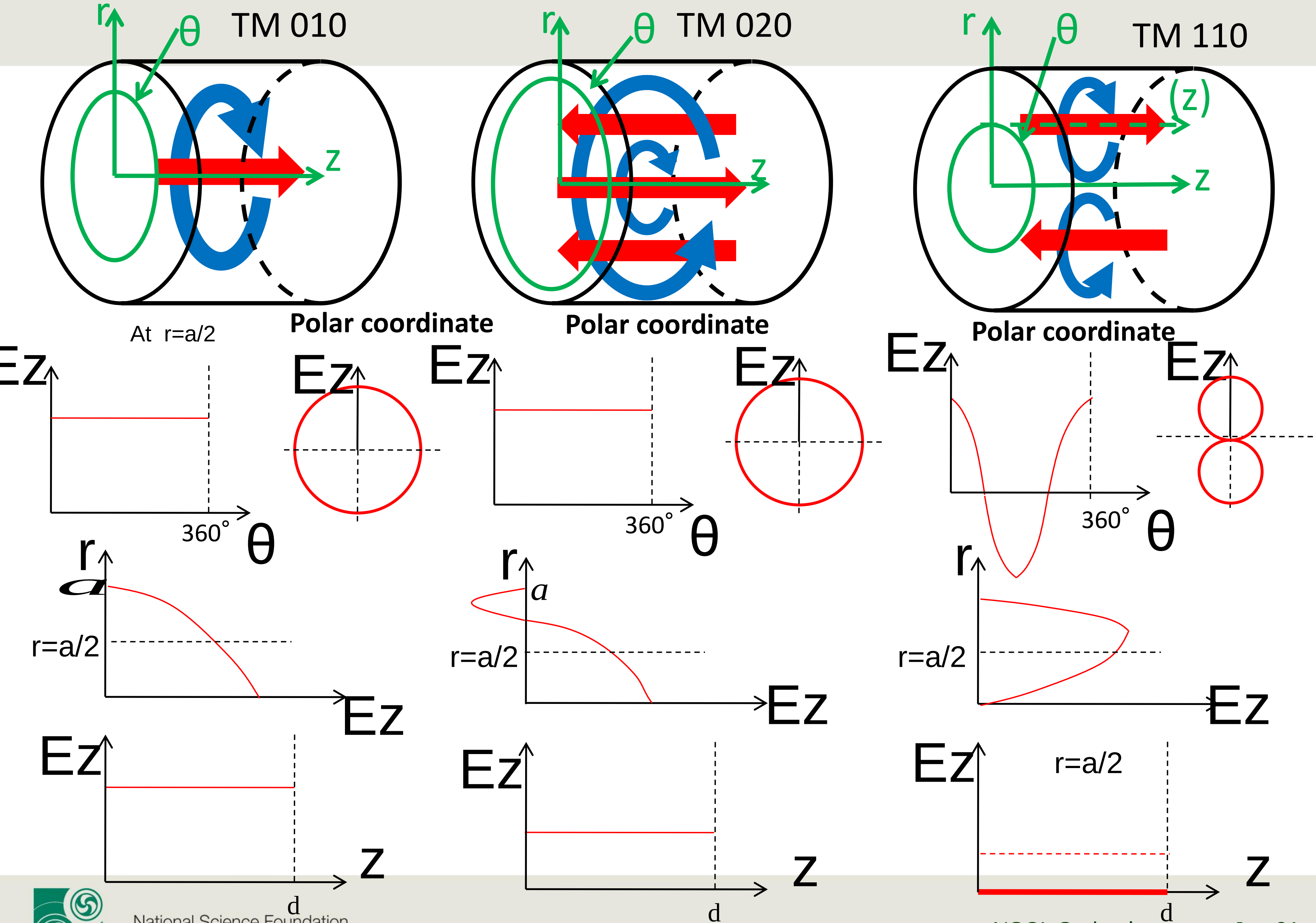
TM210

TM012

TM020

red : electric field
blue : magnetic field

Tm_{mnp} mode examples



TE Mode

Case of Ez=0, called TE-Mode, Solve Hz same as TM Mode

Again circular waveguide $$\nabla_t^2 H_z + \gamma^2 H_z = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \gamma^2 \right) H_z = 0$$

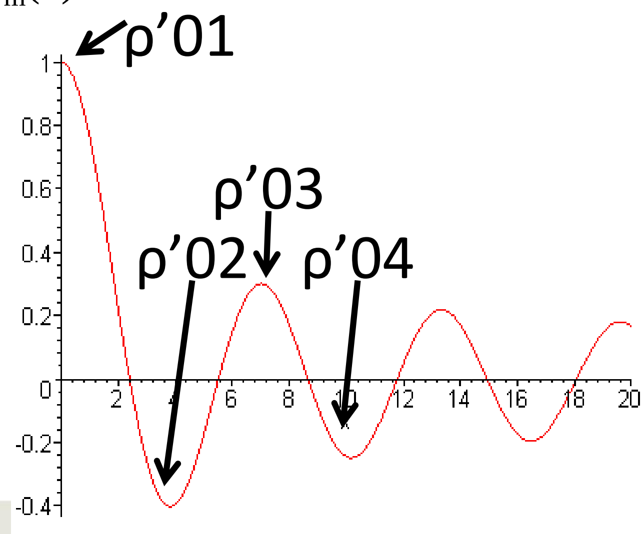
Separation of function r- component H_r and θ-component H_θ、consider plane wave same as TM Mode

$$H_z = H_0 e^{im\theta} J_m(\gamma r) e^{-i(kz-\omega t)}$$

Boundary condition: Magnetic field perpendicular to circular waveguide surface is zero .

$$H_r\Big|_{r=a} = \frac{ik}{\gamma^2}\frac{\partial H_z}{\partial r} = \frac{ik}{\gamma^2} H_0 k e^{im\theta} J_m'(\gamma r)\Big|_{r=a} = 0$$

J_m'(x): n-th zero point x=ρ_{mn}' of the deferential of the m-th Bessel function J_m(x)



$$H_z = H_0 J_m\left(\frac{\rho'_{mn}}{a} r\right) e^{im\theta} e^{-i(kz-\omega t)}$$

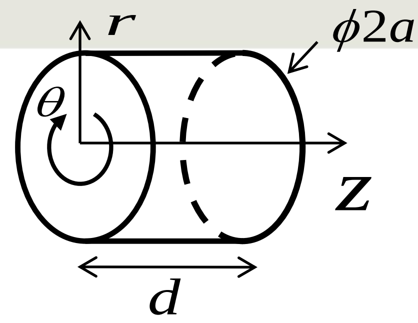
From the boundary condition of TE Mode, propagation constant γ:

$$\gamma = \frac{\rho'_{mn}}{a}$$

Field distribution of TE mode in the Pillbox

Boundary condition of TE modes: Magnetic field perpendicular to conductor surface is zero

$\Rightarrow$ $H_z = 0$ At the covers: $z = 0, d$



$$H_z = H_0 J_m\left(\frac{\rho'_{mn}}{a} r\right)\cos(m\theta)e^{-ikz} = -A\sin kz + iA\cos kz$$

Take real part $\Rightarrow$ $\mathrm{Re}[E_r(z)] = -A\sin kz$

$$\mathrm{Re}[H_z(0)] = -A\sin 0 = 0$$
$$\mathrm{Re}[E_z(d)] = -A\sin kd$$

$\rightarrow$ $\sin kd = 0 \Rightarrow kd = p\pi \Rightarrow k = \dfrac{p\pi}{d}$ (p integer)

Standing wave in z-direction,、 p is node in the z-direction
E=0 at p=0, so p > 0

TE$_{mnp}$ Modes

Real part

$$E_r = \frac{\omega\varepsilon}{\gamma^2}\frac{m}{r}E_0 J_m\left(\frac{\rho'_{mn}}{a}r\right)\sin(m\theta)\sin(\frac{p\pi z}{d})$$

$$H_r = \frac{1}{\gamma^2}\frac{p\pi}{d}E_0 J_m\left(\frac{\rho'_{mn}}{a}r\right)\cos(m\theta)\cos(\frac{p\pi z}{d})$$

$$E_\theta = \frac{\omega\varepsilon}{\gamma^2}\frac{\rho'_{mn}}{a}E_0 J_m\left(\frac{\rho'_{mn}}{a}r\right)\cos(m\theta)\sin(\frac{p\pi z}{d})$$

$$H_\theta = -\frac{1}{\gamma^2}\frac{p\pi}{d}\frac{m}{r}E_0 J_m\left(\frac{\rho'_{mn}}{a}r\right)\sin(m\theta)\cos(\frac{p\pi z}{d})$$

$$E_z = 0$$

$$H_z = E_0 J_m\left(\frac{\rho'_{mn}}{a}r\right)\cos(m\theta)\sin(\frac{p\pi z}{d})$$

Resonant frequency $\omega^2\varepsilon_0\mu_0 = \left(\dfrac{\rho'_{mn}}{a}\right)^2 + \left(\dfrac{p\pi}{d}\right)^2 \Rightarrow f = \dfrac{c}{2\pi}\sqrt{\left(\dfrac{\rho'_{mn}}{a}\right)^2 + \left(\dfrac{p\pi}{d}\right)^2}$

Cutoff frequency of TE mode in circular waveguide

k should be real number to propagate : $\vec{E} = \vec{E}_0 e^{-i(kz-\omega t)}$

Eigenvalue γ: $\gamma = \frac{\rho'_{mn}}{a}$

⇨ Condition for ω for the wave to be able to propagate in the waveguide.

$$\gamma^2 = \omega^2 \varepsilon_0 \mu_0 - k^2 \Rightarrow k^2 = \omega^2 \varepsilon_0 \mu_0 - \gamma^2 > 0 \Rightarrow \omega^2 > \frac{1}{\varepsilon_0 \mu_0} \gamma^2 = c^2 \left(\frac{\rho'_{mn}}{a} \right)^2$$

$$\Rightarrow \omega > \frac{c\rho'_{mn}}{a}$$

Cutoff frequency : $f_{Cutoff} = \frac{c\rho'_{mn}}{2\pi a}$

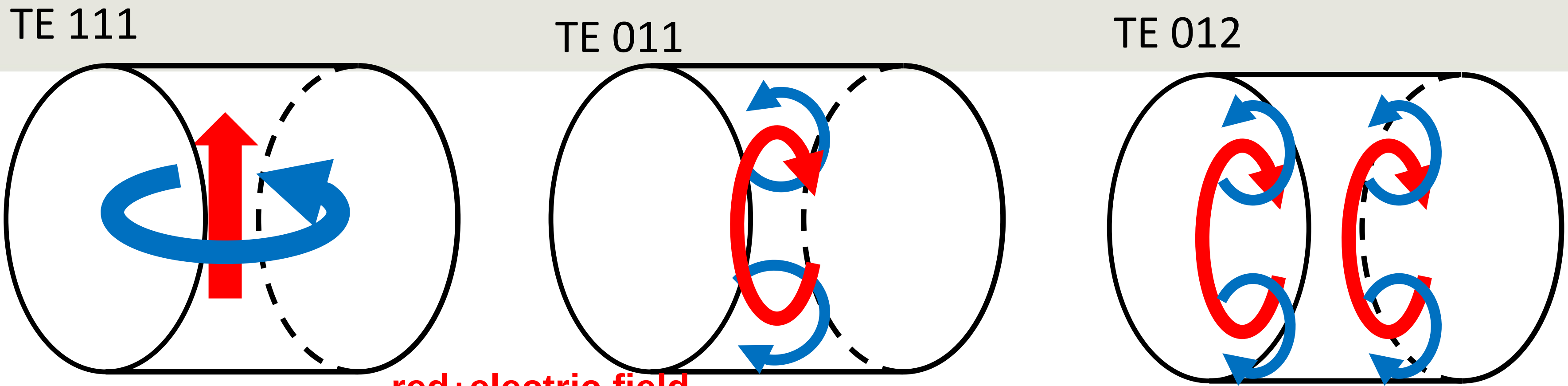
TE_{mn} mode cutoff frequency for the ϕ60mm dia. circular waveguide

Mode	Cutoff Frequency [MHz]
TE11	2928
TE21	4857
TE01	6094
TE12	8478
TE22	10665
TE02	11158

n-th zero point of $J'_m(x)$: $x=\rho'_{mn}$

m \ n	1	2	3
0	3.832	7.016	10.173
1	1.841	5.331	8.536
2	3.054	6.706	9.969

TE_{mnp} mode example



Example: the EM field distribution for TE111、TE211、TE011、TE112 in the Pillbox

Calculation of the resonant frequency of TE Modes

Comparing to the cutoff frequency with 30mm radius beam pipe, consider the field leakage into the beam pipe

$$f = \frac{c}{2\pi}\sqrt{\left(\frac{\rho'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

n-th Zeropint of $J'_m(x)$: $x=\rho'_{mn}$

n \ m	1	2	3
0	3.832	7.016	10.173
1	1.841	5.331	8.536
2	3.054	6.706	9.969

TE mnp	f [MHz]	Beampipe
TE 1 1 1	1637	Reject
TE 2 1 1	2101	Reject
TE 0 1 1	2446	Reject
TE 1 1 2	2784	Reject
TE 1 1 3	4025	propagate
TE 2 1 4	5455	propagate
TE 0 1 5	6822	propagate

Cutoff frequency of 30mm radius Beam pipe

Mode	Cutoff [MHz]
TE11	2928
TE21	4857
TE01	6094

Summary

$$\mathrm{div}\vec{D} = 0 \qquad \vec{B} = \mu_0 \vec{H}$$

$$\mathrm{div}\vec{B} = 0 \quad \mathrm{rot}\vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0} \quad \mathrm{rot}\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{0} \qquad \vec{D} = \varepsilon_0 \vec{E}$$

Summary of cylindrical cavity (pill box)

Cylinder of radius a: Rotational symmetry in Z-direction

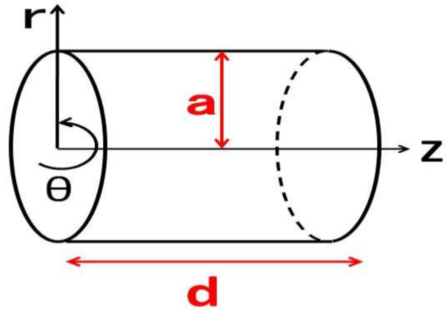
Plane wave propagates in z-direction (angular frequency ω、wave number β)

$$\begin{cases} E = E_0(x,y)e^{i(\omega t - kz)} \\ H = H_0(x,y)e^{i(\omega t - kz)} \end{cases}$$

Decomposing z, r、θ directions、put in Maxwell equations.

a=88.3mm,d=115.3mm

f=1300MHz ; ILC frequency



Et and Ht can only be **described by Ez and Hz.**

TM_{mnp} Mode (acceleration modes) : Hz=0,Ez≠0 case

$$E_z = E_0 J_m\left(\frac{\rho_{mn}}{a} r\right)\cos(m\theta)\cos\left(\frac{p\pi z}{d}\right)$$

m : number of nodes in θ direction

n : number of nodes in r direction

p : number of nodes in z direction

TE_{mnp} Mode (kick modes) : Hz≠0,Ez=0 case

$$H_z = E_0 J_m\left(\frac{\rho'_{mn}}{a} r\right)\cos(m\theta)\sin\left(\frac{p\pi z}{d}\right)$$

m : number of bellies in θ direction

n : number of bellies in r direction

p : number of bellies in z direction

7.4 Characteristic parameters of SRF cavity

1. Evacuation of RF heating on the cavity surface
2. Effective electric field electron feels
3. Parameter showing goodness of cavity shape
4. Definition of acceleration gradient
5. Relationship between surface peak electric/magnetic fields an and acceleration gradient.

Surface resistance and penetration depth in normal condor

Ampere-Maxwell law

$$\nabla\times H-\frac{\partial D}{\partial t}=J \;\Rightarrow\; \nabla\times\frac{\partial H}{\partial t}-\frac{\partial^2 D}{\partial t^2}=\frac{\partial J}{\partial t}=\sigma\frac{\partial E}{\partial t}\;(\because J=\sigma E) \;\Rightarrow\; \nabla\times\left(-\frac{1}{\mu}\nabla\times E\right)-\frac{\partial^2 D}{\partial t^2}=\frac{\partial J}{\partial t}=\sigma\frac{\partial E}{\partial t}$$

Ohm's law

$$\Rightarrow\; \Delta E=\varepsilon\mu\frac{\partial^2 E}{\partial t^2}+\sigma\mu\frac{\partial E}{\partial t}\text{ (telegraph equation)}\qquad (\because \nabla\times(\nabla\times E)=\nabla(\nabla\cdot E)-\Delta E)$$

Plane wave coming to conductor surface

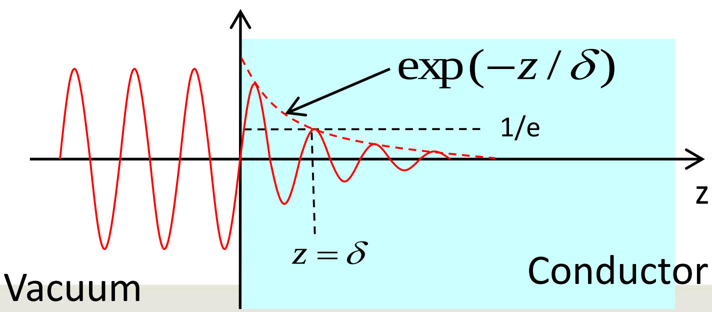
Plane wave: $E_t(z,t)=E_0e^{-i(kz-\omega t)}$ → Put telegraph equation → $\dfrac{\partial^2 E_t}{\partial z^2}=\left(-\varepsilon\mu\omega^2+i\sigma\mu\omega\right)E_t$

$$\Rightarrow\; E_t=E_0\exp\left(\sqrt{-\varepsilon\mu\omega^2+i\sigma\mu\omega}\cdot z\right)=E_0\exp\left(\sqrt{\alpha+i\beta}\cdot z\right)\;\Rightarrow\;\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}=\omega\sqrt{\frac{\mu\varepsilon}{2}\left(\pm1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right)}$$

Case of good conductors: $\dfrac{\sigma}{\varepsilon\omega}>>1$ $\quad E_t(z,t)=E_0\exp\left(-\sqrt{\dfrac{\mu\omega\sigma}{2}}z\right)\exp i\left(\omega t-\sqrt{\dfrac{\mu\omega\sigma}{2}}z\right)$

Field penetration depth (Skin Depth):

$$\boxed{\delta=\sqrt{\frac{2}{\mu\omega\sigma}}}$$



Penetration depth with normal conductors

Penetration depth(Skin Depth): $\delta = \sqrt{\dfrac{2}{\mu\omega\sigma}}$

$$E_t(z,t) = E_0 \exp\left(-\frac{z}{\delta}\right)\exp i\left(\omega t - \frac{z}{\delta}\right) = E_0 \exp\left(-\frac{1}{\delta}(1+i)z\right)\exp\left(i\omega t\right)$$

Faraday's law $\nabla \times E + \mu \dfrac{\partial H}{\partial t} = 0$

$$H_t = \frac{1}{\mu}\int\left\{\frac{1}{\delta}(1+i)E_0 \exp\left(-\frac{1}{\delta}(1+i)z\right)\exp(i\omega t)\right\}dt = \frac{1+i}{\mu\omega\delta}E_0 \exp\left(-\frac{1}{\delta}(1+i)z\right)\exp(i\omega t)$$

Surface Impedance Z: $Z \equiv \dfrac{E_t}{H_t} \equiv R_s + iX_s = \mu\omega\sqrt{\dfrac{2}{\mu\sigma\omega}}\dfrac{1}{1+i} = \mu\omega\sqrt{\dfrac{2}{\mu\sigma\omega}}\dfrac{1-i}{2} = \sqrt{\dfrac{\mu\omega}{2\sigma}}(1-i)$

Surface resistance Rs: $\mathrm{Re}(Z) = R_s = \sqrt{\dfrac{\mu\omega}{2\sigma}} = \dfrac{1}{\delta\sigma}$

How to measure the Q value in normal conductor cavities

Power loss by RF surface heating on the cavity(Wall loss) : $P_{loss} = \frac{1}{2} R_s \int |H|^2 dS$

Damping of the stored energy in the cavity per second

Energy conservation law $P_{loss} = -\frac{dU}{dt}$ Ratio of stored energy and wall loss is constant $Q = \frac{\omega U}{P_{loss}} = \frac{Const}{R_s} = \frac{\Gamma}{R_s}$

$P_{loss} = -\frac{dU}{dt} = \frac{\omega U}{Q}$ $\therefore U = U_0 e^{-\omega t/Q}$ Q is a damping constant

Consider $U(t) = \frac{\varepsilon_0}{2} |E(t)|^2$ $E(t) = E_0 e^{-\omega_0 t/2Q} e^{-i\omega t} = \int_{-\infty}^{\infty} E(\omega) e^{-i\omega t} d\omega$

Fourier transformer $E(\omega) = \frac{1}{2\pi} \int_0^{\infty} E_0 e^{-\frac{\omega_0 t}{2Q}} e^{-i\omega_0 t} e^{i\omega t} dt = \frac{1}{2\pi} \int_0^{\infty} E_0 \exp(-\frac{\omega_0 t}{2Q} - i\omega_0 t + i\omega t) dt = \frac{1}{2\pi} \frac{E_0}{-\frac{\omega_0}{2Q} + i(\omega - \omega_0)}$

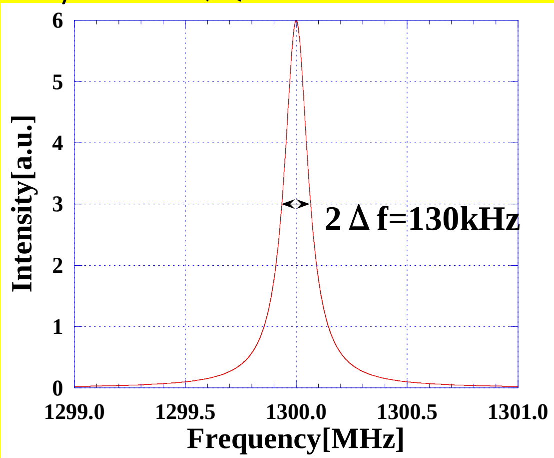
In power expression , Lorenz curve

$$P = |E(\omega)|^2 = \left| \frac{A}{-\frac{\omega_0}{2Q} + i(\omega - \omega_0)} \right|^2 = \frac{A^2}{\left(\frac{\omega_0}{2Q}\right)^2 + (\omega - \omega_0)^2}$$

$$P = \frac{A^2}{2} = \frac{A^2}{\left(\frac{\omega_0}{2Q}\right)^2 + (\omega - \omega_0)^2}$$

$$\boxed{Q = \frac{f_0}{2\Delta f}} \quad (\because \omega_0 = 2\pi f_0)$$

Example: resonant frequency1300MHz、Q=10000



Loaded Q (actual Q): Q_L

- Unloaded Q (Q_O) is the intrinsic Q of the cavity, which relates to cavity itself.
  - NC case, Qo ~ 30000 at 1.3GHz Δf ~ 22 kHz
  - Qo is very large, for instance:
    - » Qo ≈ 2 x 10^{10}
    - » Bandwidth Δf = f/(2xQo) = 1.3x10^9/(2x 2x10^{10}) = 0.033 Hz
  - No way to measure at cold ??

- Loaded Q (Q_L): actual cavity Q value including RF coupler (P_e), RF pickup (P_t), and beam loading (P_b) .

$$Q_L = \frac{\omega U}{P_{loss} + P_t + P_e + Pb}, \qquad Pb = I\mathrm{x}V$$

$$= \frac{f_0}{2\Delta f}, \quad no\, beam$$

Cavity performance parameter (Pillbox case)

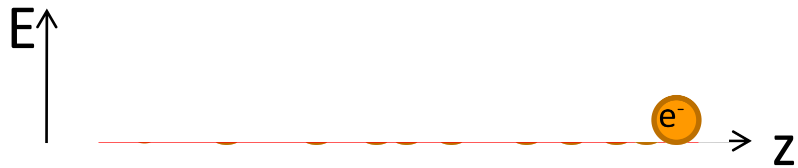
Accelerating Voltage: Effective accelerating voltage when electron passes the cavity

When electron passes on the beam axis, particle location is z=ct   Pillbox TM010 case $k = \frac{p\pi}{d} = 0$

$$V_c = \left|\int_0^{L_{eff}} E_0 \exp\{i(\omega t - kz)\}dz\right| = \left|\int_0^{L_{eff}} E_0 \exp\left\{i\frac{\omega}{c}z\right\}dz\right| = \left|\left[-\frac{iE_0}{\omega/c}\exp\left\{i\left(\frac{\omega}{c}\right)z\right\}\right]_0^d\right| = \frac{2E_0}{\omega/c} = \frac{2E_0 d}{\pi}$$

$$\gamma^2 = (\omega/c)^2 - k^2 \Rightarrow (\omega/c)^2 = (\rho_{01}/a)^2 = \left(\frac{\rho_{01}}{\rho_{01}d/\pi}\right)^2 = \left(\frac{\pi}{d}\right)^2$$

$$\gamma = \rho_{01}/a \qquad a = \frac{\rho_{01}}{\pi}d$$



Transit-time factor : Time effect particle passed the time varied field

$$T = \frac{V_c}{\left|\int_0^d E_0 dz\right|} = \frac{V_c}{E_0 d} = \frac{2}{\pi} = 0.64$$

Effective accelerating gradient Eacc: Electric filed gradient for electron actually to feel when electron pass the cavity

$$E_{acc} = \frac{V_c}{L_{eff}} = \frac{dE_0 T}{d} = E_0 T = \frac{2E_0}{\pi} = 0.64\,[MV/m]$$

Stored Energy : Energy stored in the cavity

$$U = \frac{\mu_0}{2}\int |H|^2 dV = \frac{\varepsilon_0}{2}\int |E|^2 dV = \frac{\pi\varepsilon_0 d a^2 E_0^2 J_1^2(\rho_{01})}{2} = 3.38\times10^{-3}[J]$$

Wall loss : RF heating loss on the cavity wall

$$P_{loss} = \frac{R_s}{2}\int |H_s|^2 dS = \frac{R_s \pi\varepsilon_0 a(a+d)E_0^2 J_1^2(\rho_{01})}{\mu_0} = 1012.5 \quad [W]$$

Calculate the parameters under the given condition

$E_0 = 1.0\,[MV/m]@1300MHz$

$\varepsilon_0 = 8.85418782\times10^{-12}[F/m]$

$\mu_0 = 1.2566370614\times10^{-6}[H/m]$

$c = 2.99792458\times10^{8}[m/s]$

$R_s = 9.4067\times10^{-3}[Ohm](20^{O}C, \text{Copper})$

$J_1(\rho_{01}) = 0.519$

$a = 0.0883[m] \quad d = 0.1153[m]$

Cavity performance parameter (Pillbox case) continued

Unloaded quality factor (Q_0) : Intrinsic Q value of the cavity

$$Q_0 = \frac{\omega U}{P_{loss}} = \frac{\omega \frac{\varepsilon_0}{2}\int_0^d |E|^2 dV}{\frac{R_s}{2}\int_S |H|^2 dS} = \frac{\omega \mu_0 da}{2R_S(a+d)} = 2.7\times 10^4$$

$Q_0 \propto 1/\sqrt{\omega}$

$Q_0 \propto \sqrt{\sigma}$

Geometry factor : ratio of stored energy and wall loss, goodness of cavity shape

$$\Gamma = Q_0 \cdot R_S = \frac{\omega \mu_0 da}{2(a+d)} = 256\,[Ohm]$$

Independent on ω $\because d, a \propto \frac{1}{\omega}$

Shunt impedance : Parameter shows how much energy are concentrated on beam, acceleration efficiency

$$R_{sh} = \frac{V_c^{\ 2}}{P_{\ loss}} = \frac{\mu_0\left(\frac{2d}{\pi}\right)^2}{R_S \pi \varepsilon_0 a(a+d) J_1^2(j_{01})} = \quad 5.3 \quad [MOhm]$$

Depends on shape and material

$R_{sh} \propto \frac{1}{\sqrt{\omega}}$

$R_{sh} \propto \frac{1}{R_S}$

R/Q : Parameter of acceleration efficiency

$$\frac{R_{sh}}{Q_0} = \frac{V_c^2}{P_{loss}}\frac{P_{loss}}{\omega U} = \frac{V_c^2}{\omega U} = \frac{8}{\pi^2 \varepsilon_0 c \rho_{01}^2 J_1^2(\rho_{01})} = 195 \quad [Ohm]$$

Independent on ω

Maximum surface electric/magnetic field ratio on Eacc

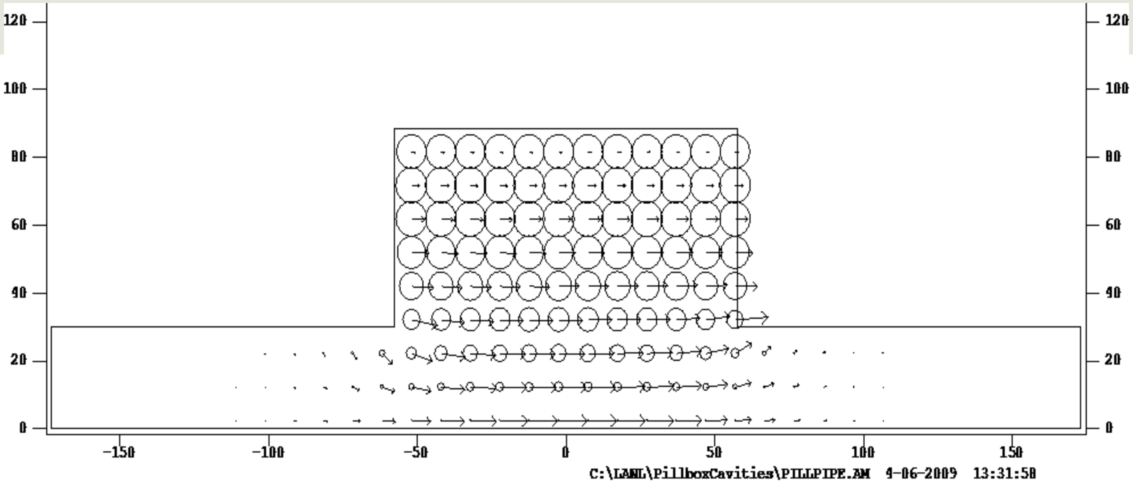
$$\frac{E_p}{E_{acc}} = \frac{\pi}{2} \qquad \frac{H_p}{E_{acc}} = \frac{\pi}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} J_1\left(\frac{\rho_{01}}{a} r\Big|_{sp}\right) = 30.5 \quad [Oe/(MV/m)]$$

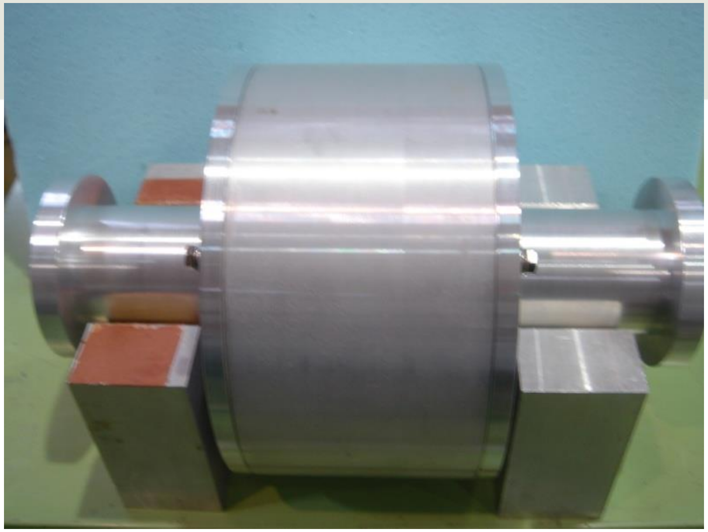
Critical field of Nb:1800[Oe]

$$1[Oe] = \frac{1}{4\pi}\times 10^3 [A/m]$$

Oe = Gauss= 10^{-4} T

Pillbox cavity design with Φ60mm beam pipe using design code (Superfhish)



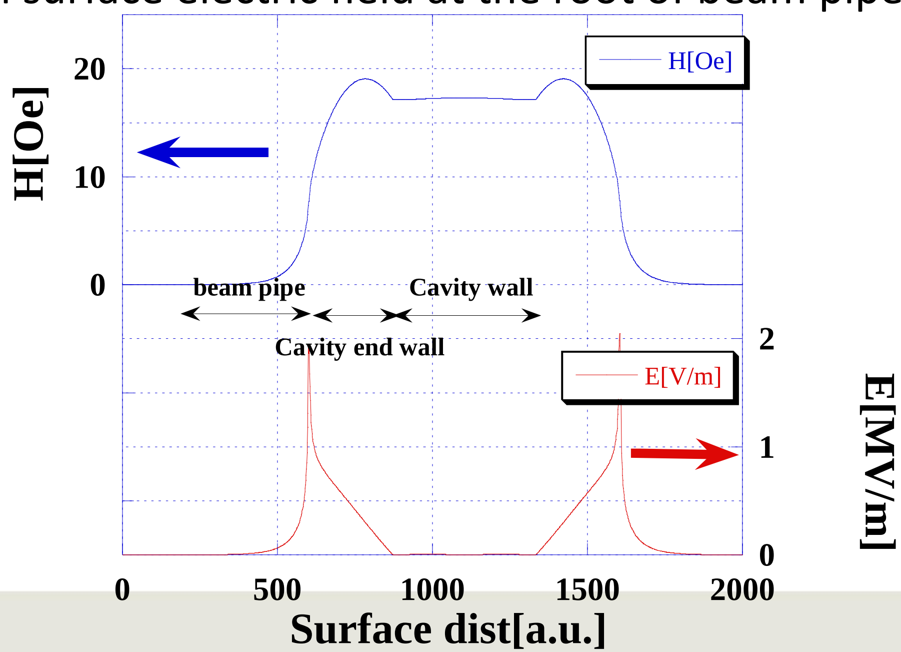


Bore diameter[mm]	60
Frequency[MHz]	1321
Transit-time factor	0.51
Geometrical factor[Ohm]	263.0
R/Q[Ohm]	126.8
Hp/Eacc[Oe/(MV/m)]	37.6
Ep/Eacc	4.04

These parameters are especially important for SRF cavity design.

Surface magnetic field is high.

High surface electric field at the root of beam pipe.



Frequency dependence of the cavity RF parameters

Characteristics Parameter	Normal conducting ω dependence	Superconducting ω dependence
R_S [Ω]	$\omega^{1/2}$	ω^{2}
P_{loss} [W]	$\omega^{-3/2}$	No depend
U [J]	ω^{-3}	ω^{-3}
Q_O	$\omega^{-1/2}$	ω^{-2}
Rsh [Ω]	$\omega^{-1/2}$	ω^{-2}
Rsh/L [W/m]	$\omega^{-1/2}$	ω^{-1}
Γ [Ω]	No depend	No depend
R/Q [Ω]	No depend	No depend

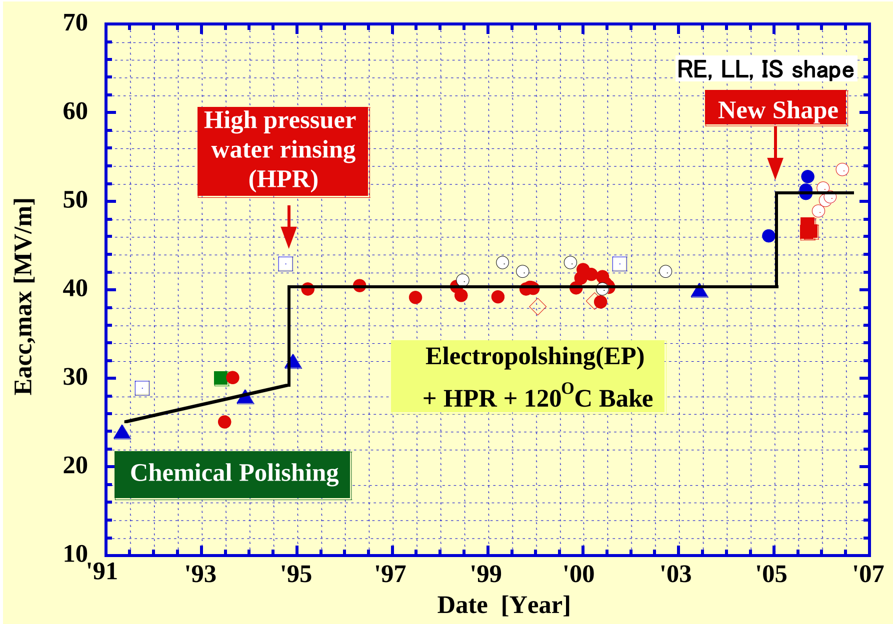
Rsh per unit meter is proportional to $\sqrt{\omega}$

Which is a reason that higher frequency is preferred in normal conducting linear collider (11.4 GHz)

7.5 Criteria for cavity shape

- Suppressed Multipacting
- Lower Surface Electric field against field emission (smaller Ep/Eacc)

2001 - 2004



- Lower Surface Magnetic field (smaller Bp/Eacc)
- High Efficient
- High Gradient

incorporate

- Real completed cavity design uses computer codes: Superfish (2D), Sold works (3D)…….

7.6 Real cavity design

General trends

Diameter of BP or bore size ➡ bigger

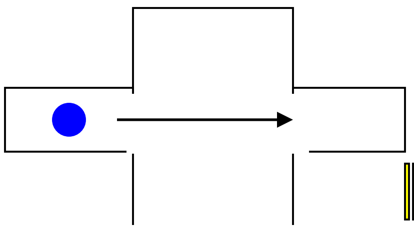
- **Ep/Eacc ⟶ larger**
- **Bp/Eacc ⟶ larger**
- **R/Q ⟶ smaller**
- **Cell to cell coupling ⟶ lager**
- **HOM issue ⟶ more relaxed**

1) Need a aperture on the cavity for charged particles to pass
2) Need RF input port to feed RF power
3) HOM coupler port to extract HOM power

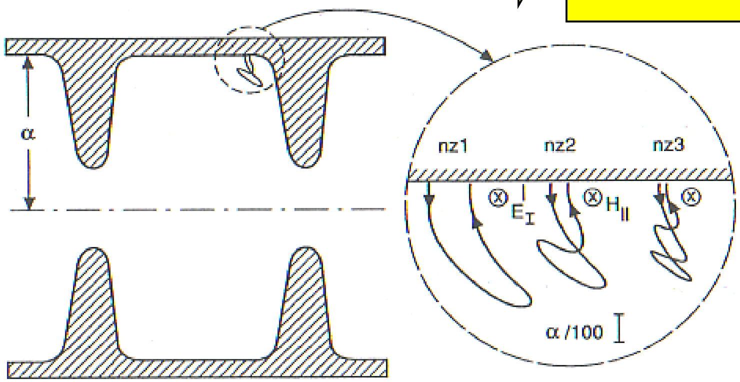
Need Beam pipes on both Ends

4) High efficient cavity
5) Better performance
Smaller Ep/Eacc : Multipacting, Field emission
Smaller Bp/Eacc : Fundamental limit
6) Bore size impacts on the performance

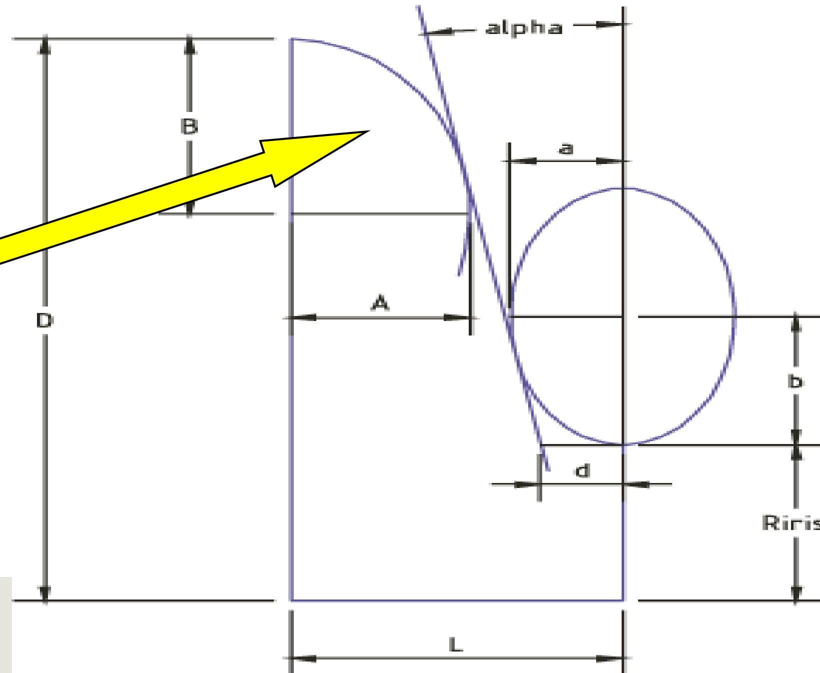
Optimization of cell shape
Multi-cell cavity



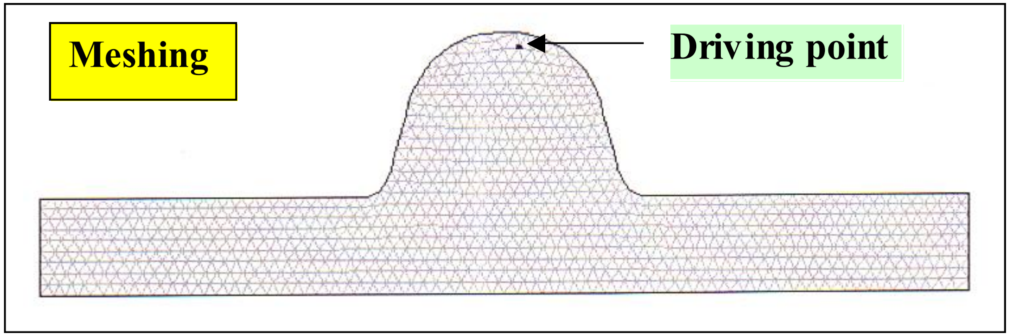
Choose spherical shape to reduce multipacting

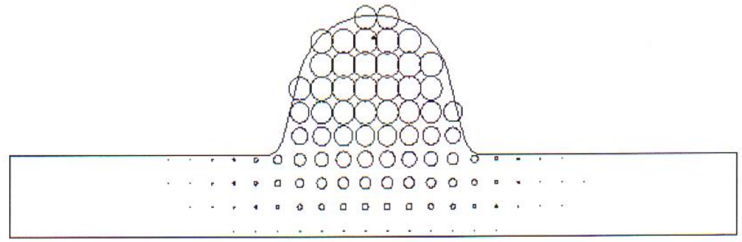


Stanford university

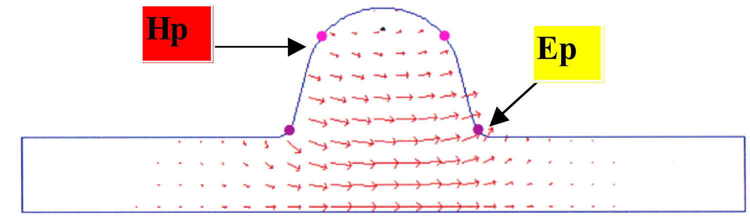


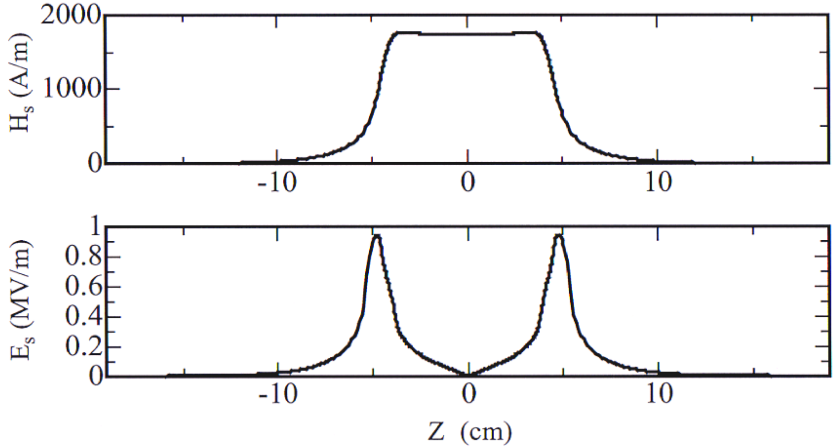
SRF single cell cavity design by Superfish, example





All calculated values below refer to the mesh geometry only.
Field normalization (NORM = 0): EZERO = 1.00000 MV/m
Length used for E0 normalization = 10.76000 cm
Frequency (starting value = 1300.000) = 1293.77430 MHz
Particle rest mass energy = 0.510999 MeV
Beta = 1.0000000
Normalization factor for E0 = 1.000 MV/m = 7048.913
Transit-time factor Abs(T+iS) = 0.5454664
Stored energy = 0.0038869 Joules
Using standard room-temperature copper.
Surface resistance = 9.38405 milliOhm
Normal-conductor resistivity = 1.72410 microOhm-cm
Operating temperature = 20.0000 C
Power dissipation = 1118.1551 W
Q = 28257.6 Shunt impedance = 96.230 MOhm/m
Rs*Q = 265.171 Ohm Z*T*T = 28.632 MOhm/m
r/Q = 109.024 Ohm Wake loss parameter = 0.22157 V/pC
Average magnetic field on the outer wall = 1729.9 A/m, 1.40411 W/cm^2
Maximum H (at Z,R = 3.32643,8.55466) = 1753.44 A/m, 1.44258 W/cm^2
Maximum E (at Z,R = 4.75232,4.24425) = 0.946176 MV/m, 0.02953 Kilp.
Ratio of peak fields Bmax/Emax = 2.3288 mT/(MV/m)
Peak-to-average ratio Emax/E0 = 0.9462





Example of calculation of final characteristic cavity parameters

Homework 3

In case of single cell cavity with beam tube, definition acceleration field becomes more questionable due to the field leaking into the tube.

We define the effective cavity length as βλ/2 no concerning to the real cell length.

Superfish outputs

f=1293.77430 MHz

Ploss=1118.1551 W

Rs・Q=265.171 Ohm

Qo=28257.6

(Rsh/Q)=109.24 Ohm

Hs,max=1753.44 A/m

Es,max=0.946176 MV/m

Homework

Calcurate bellow parmeter using the left data。

Rsh =

$V =$

$\lambda =$

Eacc = V/Leff **effective length: λ/2**

Bp/Eacc = mT/(MV/m), **1[A/m] = $4\pi \times 10^{-7}$ [T]**

Ep/Eacc =

Eacc $= C \cdot \sqrt{P_{loss} \cdot Q_O}$,

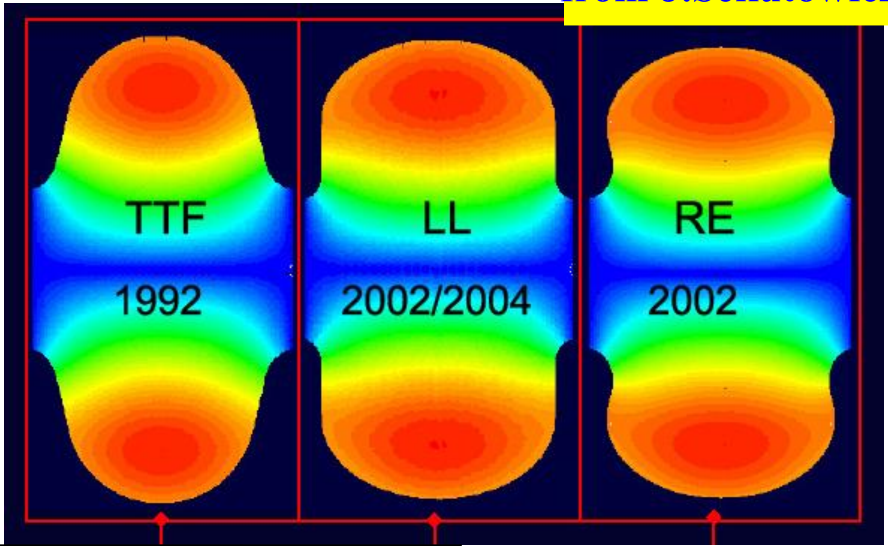
C =

7.7 High gradient shapes

Cavity shape designs with low Bp/Eacc

from J.Sekutowicz lecture Note

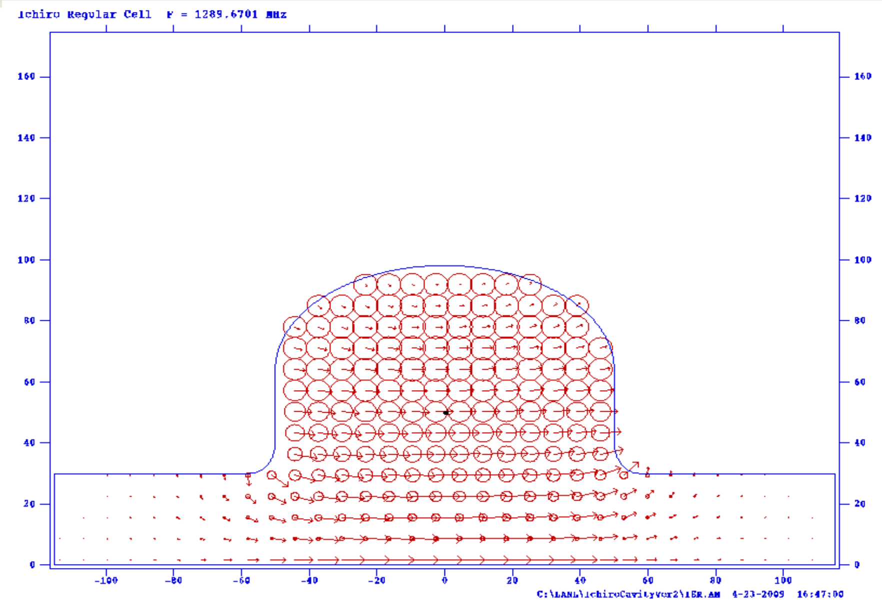
TTF: TESLA shape
Reentrant (RE): Cornell Univ.
Low Loss(LL): JLAB/DESY
Ichiro‐Single（IS）： KEK

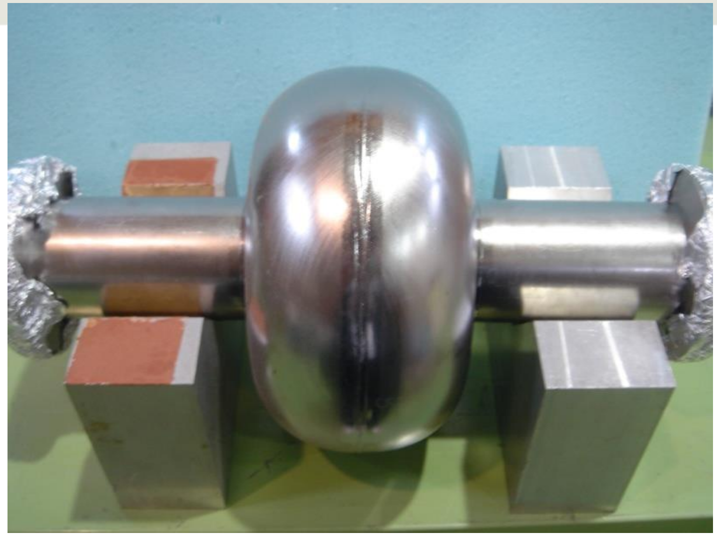


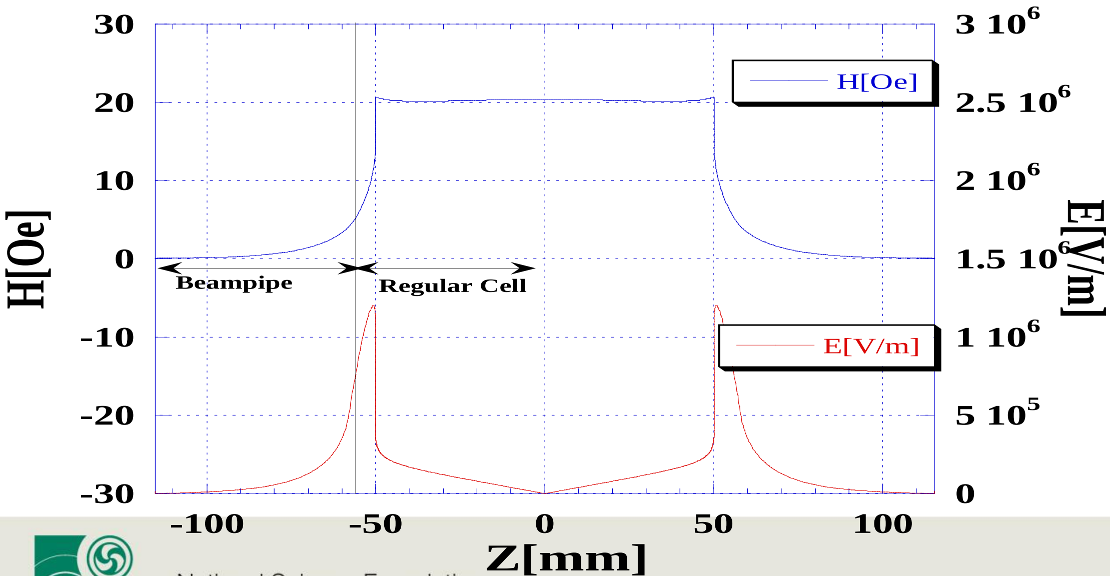
	TESLA	LL	RE	IS
Diameter [mm]	**70**	**60**	**66**	**61**
Ep/Eacc	**2.0**	**2.36**	**2.21**	**2.02**
Bp/Eacc [Oe/MV/m]	**42.6**	**36.1**	**37.6**	**35.6**
R/Q [W]	**113.8**	**133.7**	**126.8**	**138**
G[W]	**271**	**284**	**277**	**285**
Eacc max	**41.1**	**48.5**	**46.5**	**49.2**

Cavity design of Ichiro single cell(Regular Cell)

Design cavity with suppressed surface magnetic field at 1.3GHz

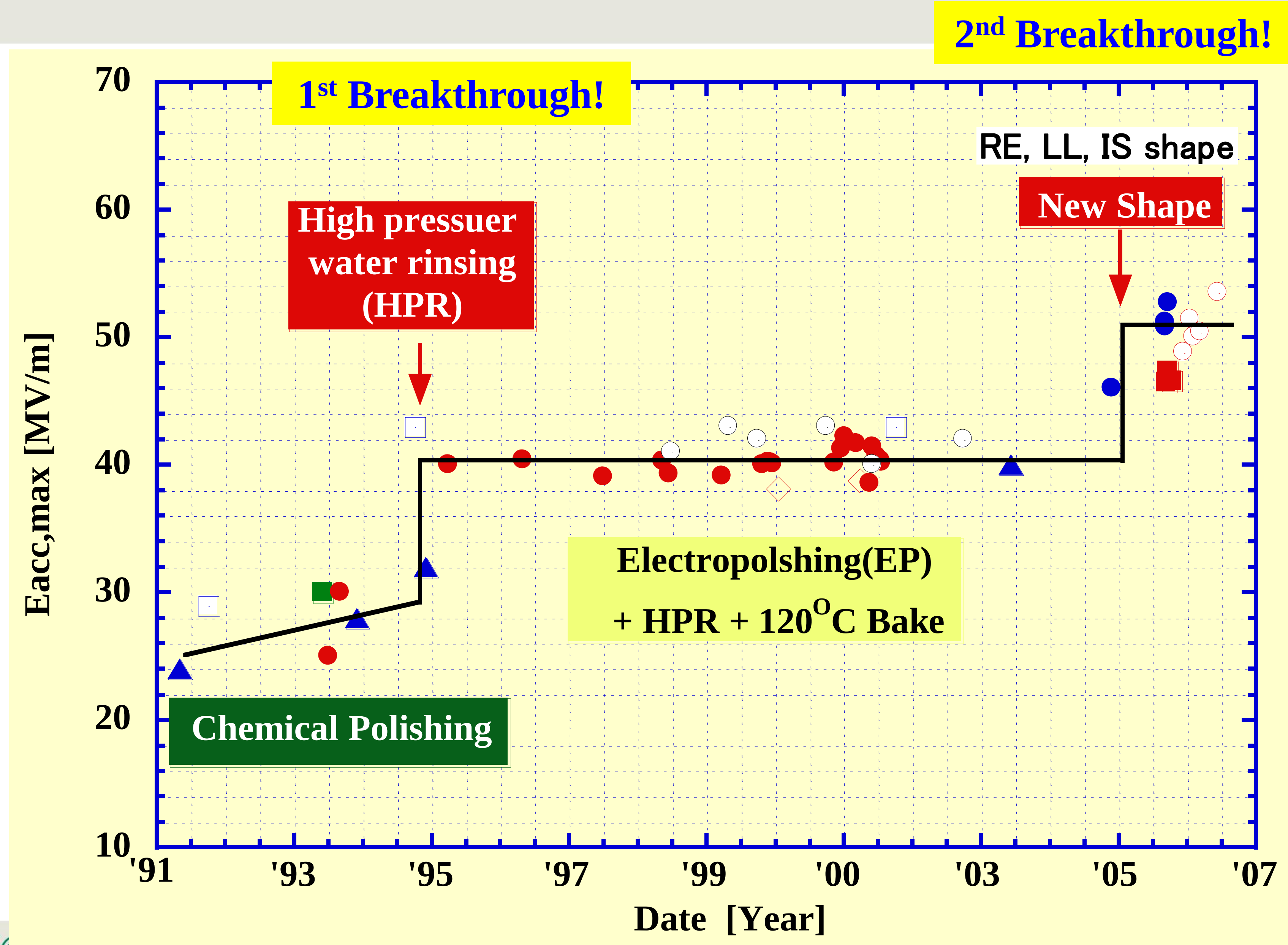






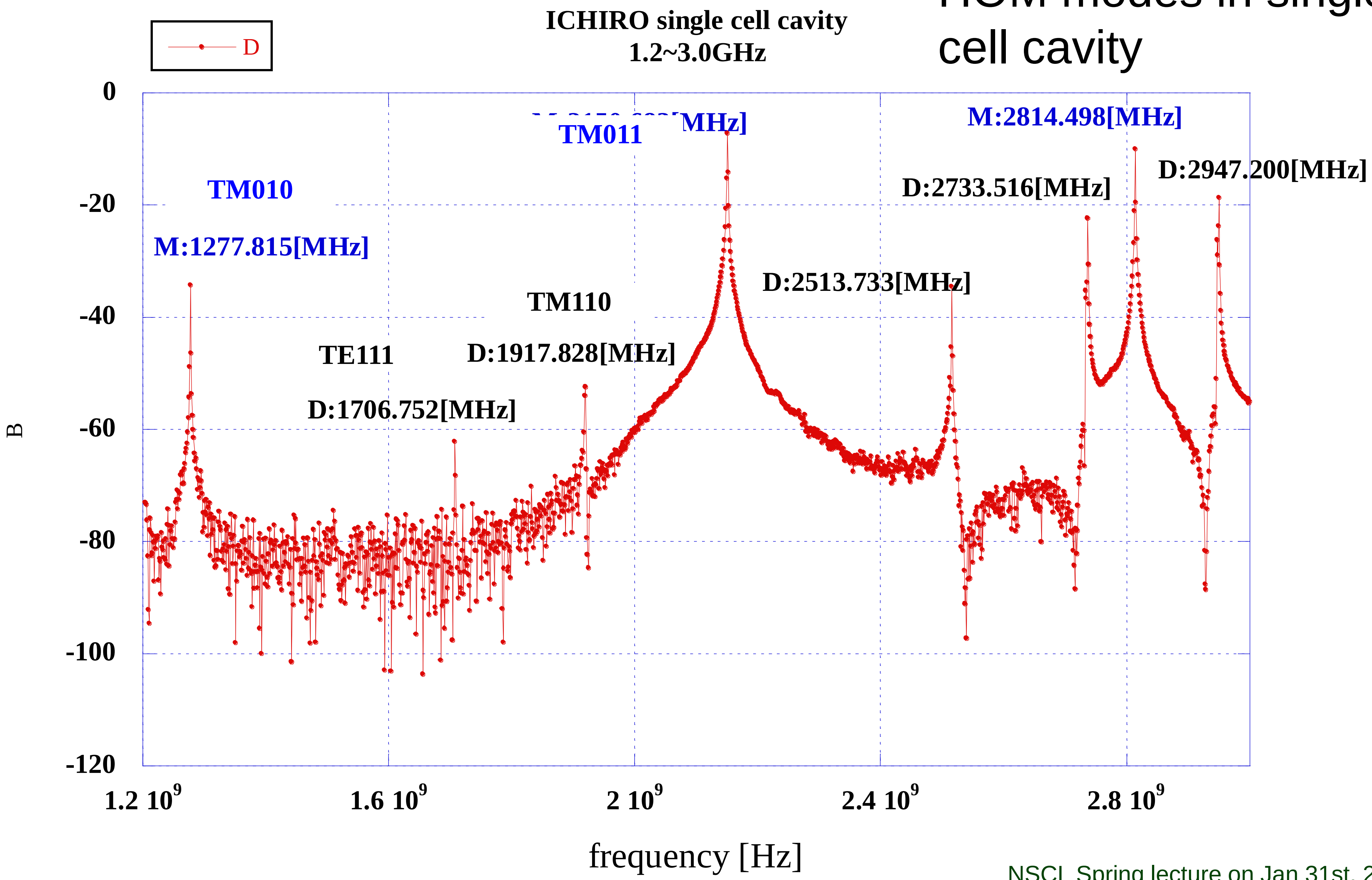
Ichiro Regular Cell	
Frequency[MHz]	1289.6701
Transit-time factor	0.5803802
Stored energy[J]	0.0039884
Surface resistance[nOhm]	26.5302
Wall loss[μW]	3042.4839
Unloaded Q	1.06E+10
Geometrical factor[Ohm]	281.819
R/Q[Ohm]	138.566
Shunt impedance[GOhm]	1471.9709
Hp[A/m]	20.644933
Bp/Eacc[Oe/(MV/m)]	35.571394
Ep/Eacc	2.0718832
EaccMax	50.602459

Eacc vs. Year


2nd Breakthrough!
1st Breakthrough!
RE, LL, IS shape
New Shape
High pressuer water rinsing (HPR)
Electropolshing(EP)
+ HPR + 120OC Bake
Chemical Polishing
Eacc,max [MV/m]
10
20
30
40
50
60
70
'91
'93
'95
'97
'99
'00
'03
'05
'07
Date [Year]

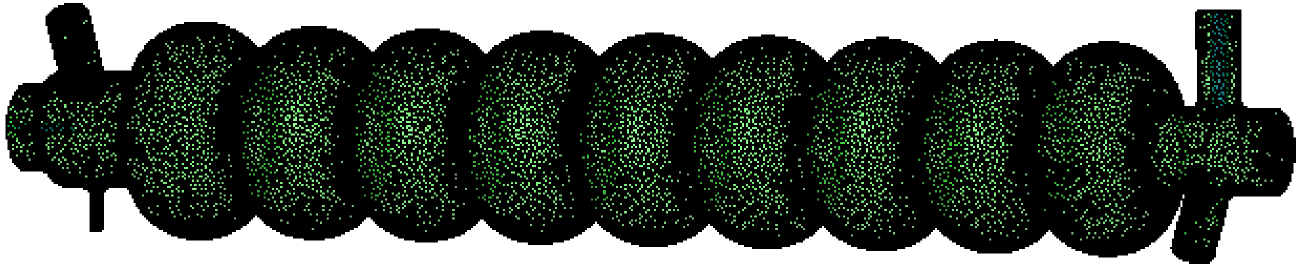
7.8 Higher order modes

HOM modes in single cell cavity



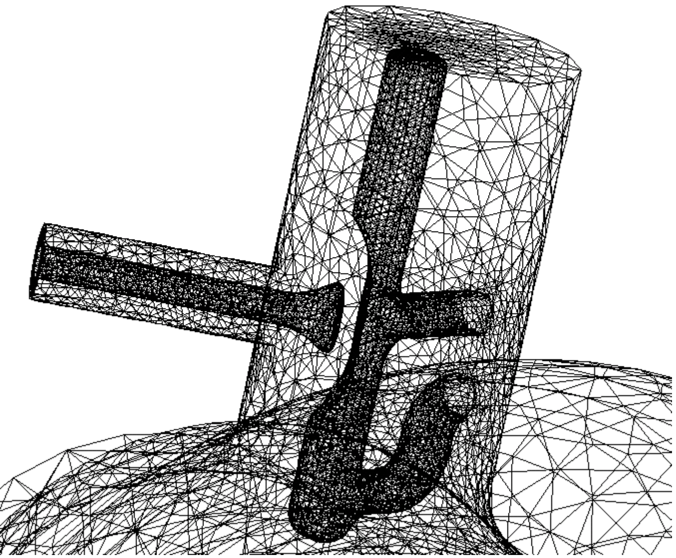
7.9 Multi-cell cavity design

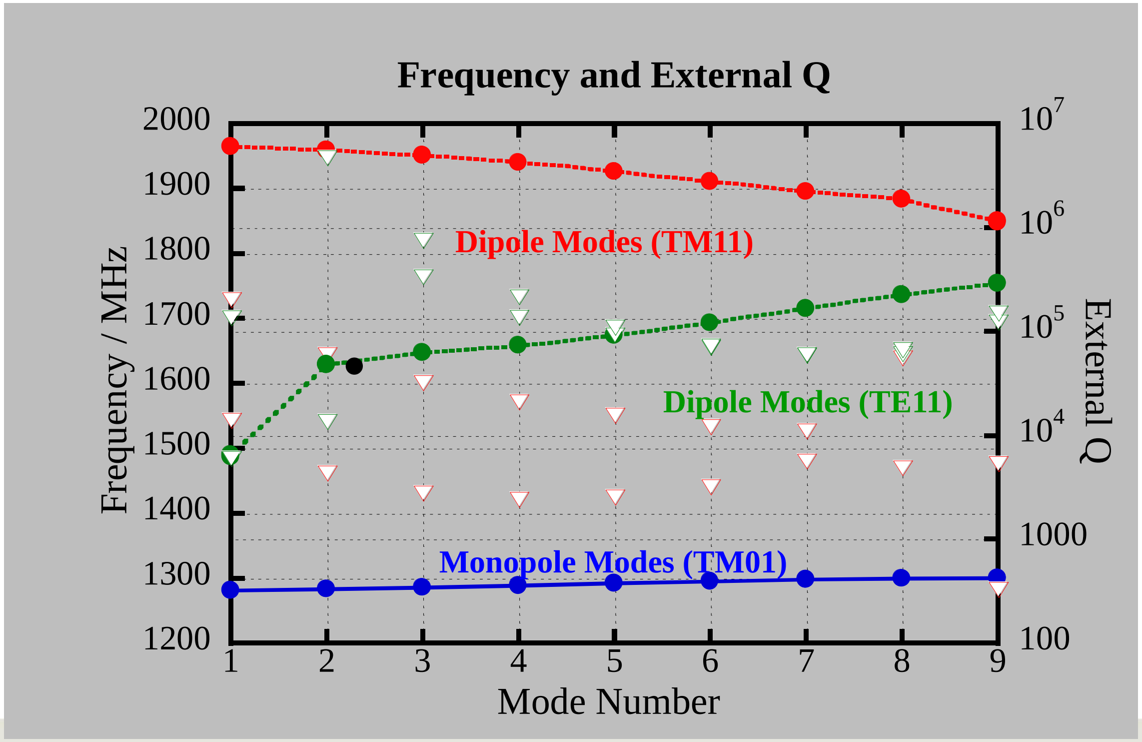
Element Model (Mesh)



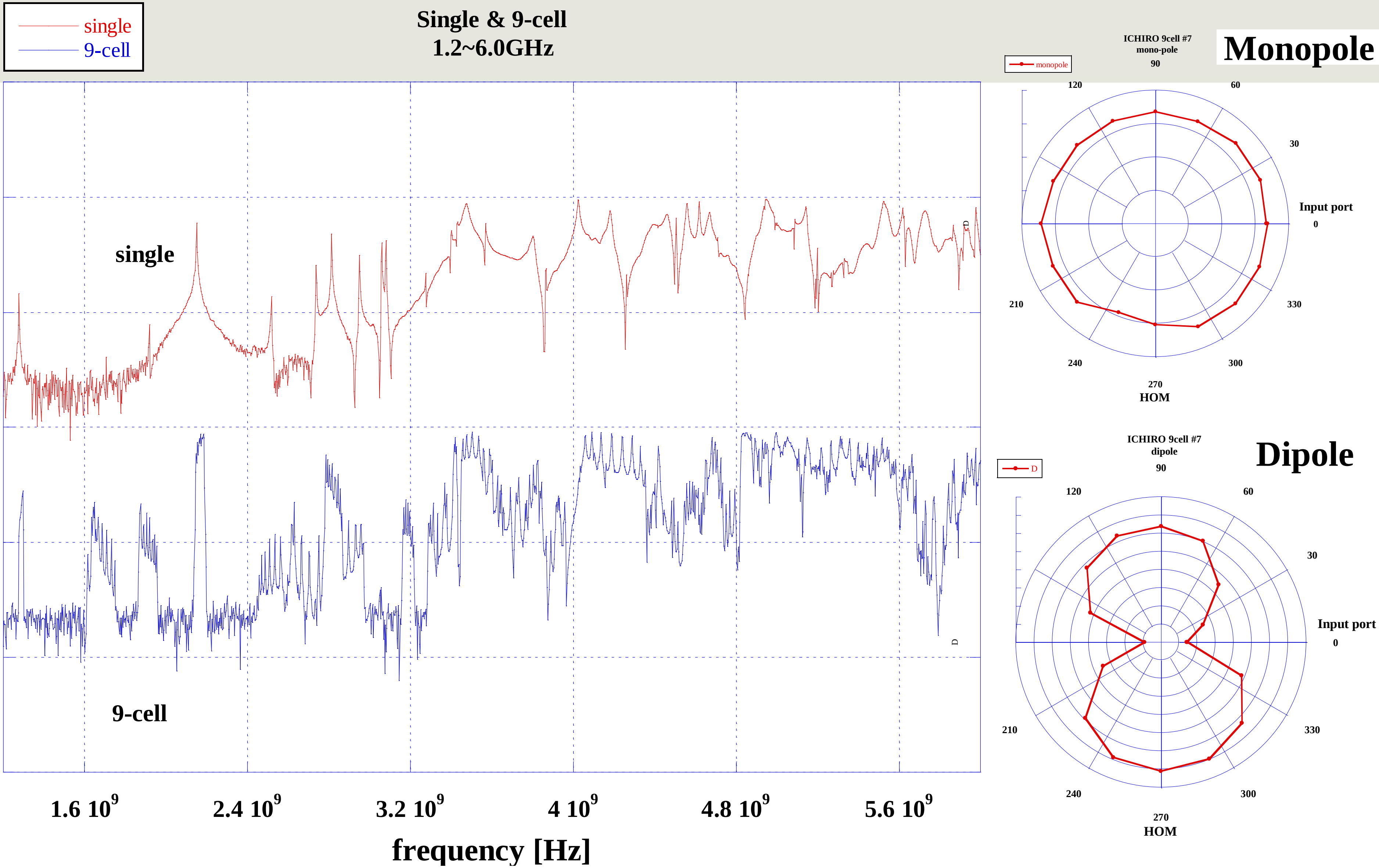
Simulation of Higher Order Mode Damping

Mesh of HOM Damper



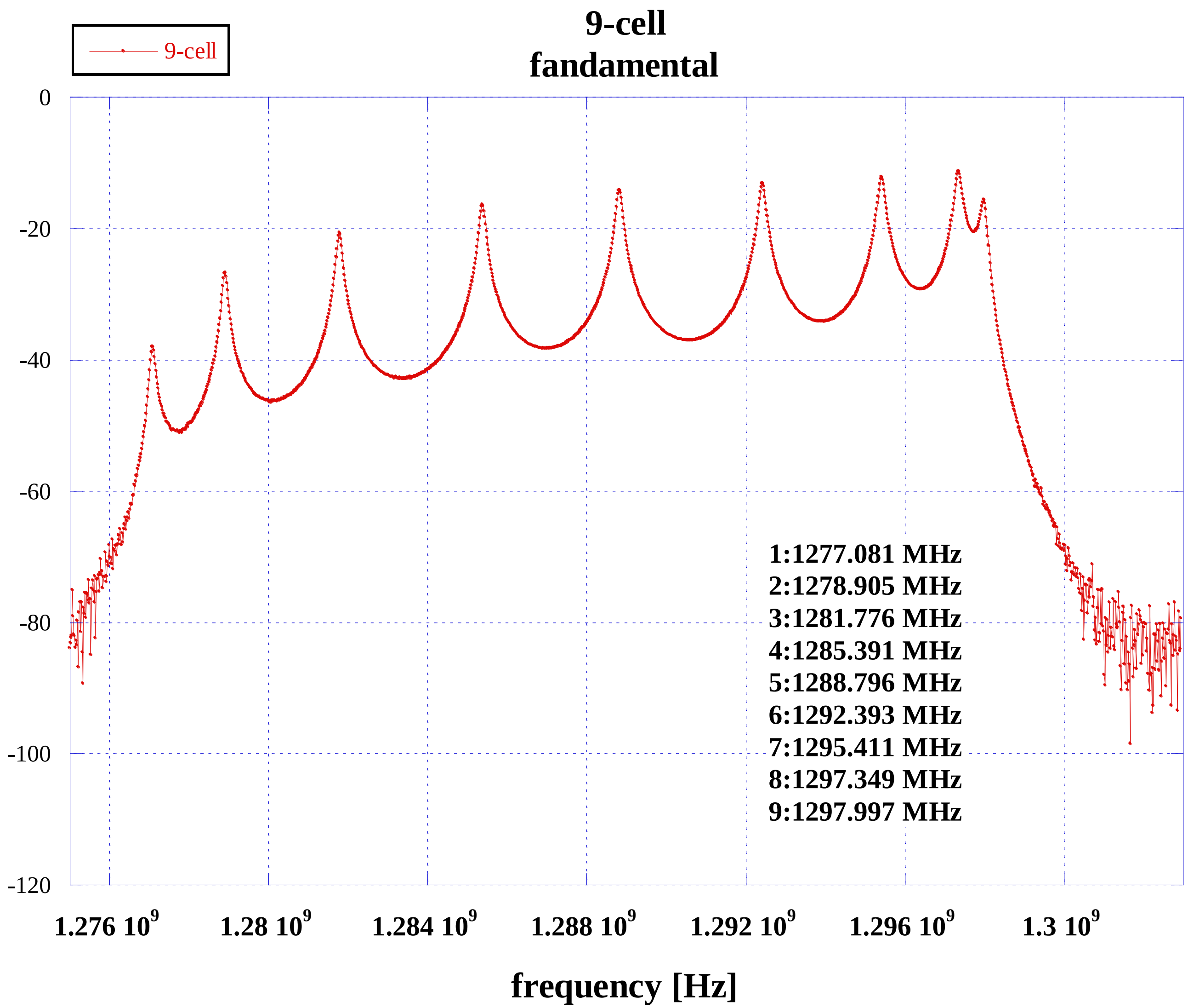


Comparison of HOM modes between Single cell and 9-cell cavity



Each eigen-mode of single cell split into 9 in a 9-cell structure.

Pass band in Acceleration modes
TM_{010} in a 9-cell cavity



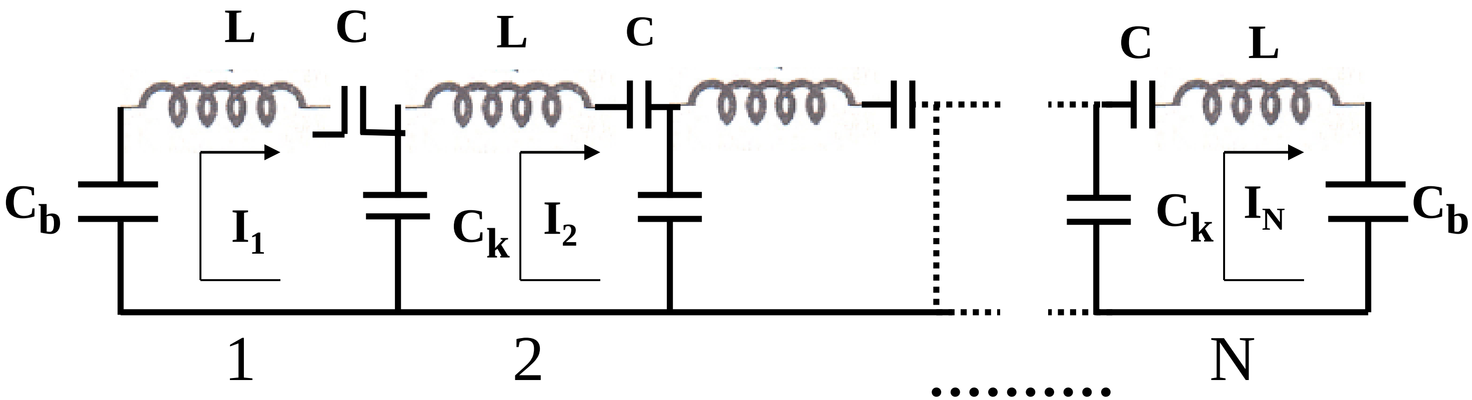
Criteria for multi-cell structures

Pros and cons for multi-cell structures

- *Cost of accelerators is lower (less auxiliaries: LHe vessels, tuners, fundamental power couplers, control electronics)*
- *Higher real-estate gradient (better fill factor)*

- *Field flatness vs. N*
- *HOM trapping vs. N*
- *Power capability of fundamental power couplers vs. N*
- *Chemical treatment and final preparation become more complicated*
- *The worst performing cell limits whole multi-cell structure*

Equivalent circuit model for multi-cell cavity



$$\left(\frac{1}{i\omega C_b}+i\omega L\right)I_1+\left(\frac{1}{i\omega C}\right)I_1+\left(\frac{1}{i\omega C_k}\right)(I_1-I_2)=0$$

$$\frac{1}{i\omega C_k}(I_j-I_{j-1})+\left(i\omega L+\frac{1}{i\omega C}\right)I_j+\left(\frac{1}{i\omega C_k}\right)(I_j-I_{j+1})=0$$

$$1<j<N$$

$$\left(\frac{1}{i\omega C_k}\right)(I_N-I_{N-1})+\left(i\omega L+\frac{1}{i\omega C}\right)I_N+\left(\frac{1}{i\omega C_b}\right)I_N=0$$

$$(1+k+\gamma)I_1-kI_2=\Omega I_1$$

$$-kI_{j-1}+(1+2k)I_j-kI_{j+1}=\Omega I_j,\quad 1<j<N$$

$$-kI_{N-1}+(1+k+\gamma)I_N=\Omega I_N$$

$$\omega_0^2=\frac{1}{LC},\ k=\frac{C}{C_k},\ \gamma=\frac{C}{C_b},\ \text{and}\ \Omega=\frac{\omega^2}{\omega_0^2}$$

$$\begin{bmatrix} 1+k+\gamma & -k & 0 & \cdots & 0 \\ -k & 1+2k & -k & \cdots & 0 \\ 0 & & \ddots & & 0 \\ \vdots & & -k & 1+2k & -k \\ 0 & \cdots & 0 & -k & 1+k+\gamma \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{N-1} \\ I_N \end{bmatrix}=\Omega\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{N-1} \\ I_N \end{bmatrix}$$

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Solution for j-th cell in m-th mode

π-mode

$$\mathrm{v} = \frac{1}{\sqrt{N}}\left[1,\ -1,\ 1,\ -1,\ \cdots\right]$$

$$\begin{bmatrix} 1+k+\gamma & -k & 0 & \cdots & 0 \\ -k & 1+2k & -k & \cdots & 0 \\ 0 & & \ddots & & 0 \\ \vdots & & -k & 1+2k & -k \\ 0 & \cdots & 0 & -k & 1+k+\gamma \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ \vdots \\ \\ \\ \end{bmatrix} = \Omega^{(N)} \begin{bmatrix} 1 \\ -1 \\ \vdots \\ \\ \\ \end{bmatrix} \Rightarrow$$

General equation

$$\begin{bmatrix} 1+3k & -k & 0 & \cdots & 0 \\ -k & 1+2k & -k & \cdots & 0 \\ 0 & & \ddots & & 0 \\ \vdots & & -k & 1+2k & -k \\ 0 & \cdots & 0 & -k & 1+3k \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{N-1} \\ I_N \end{bmatrix} = \Omega \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{N-1} \\ I_N \end{bmatrix}$$

$$1+2k+\gamma = \Omega^{(N)}$$

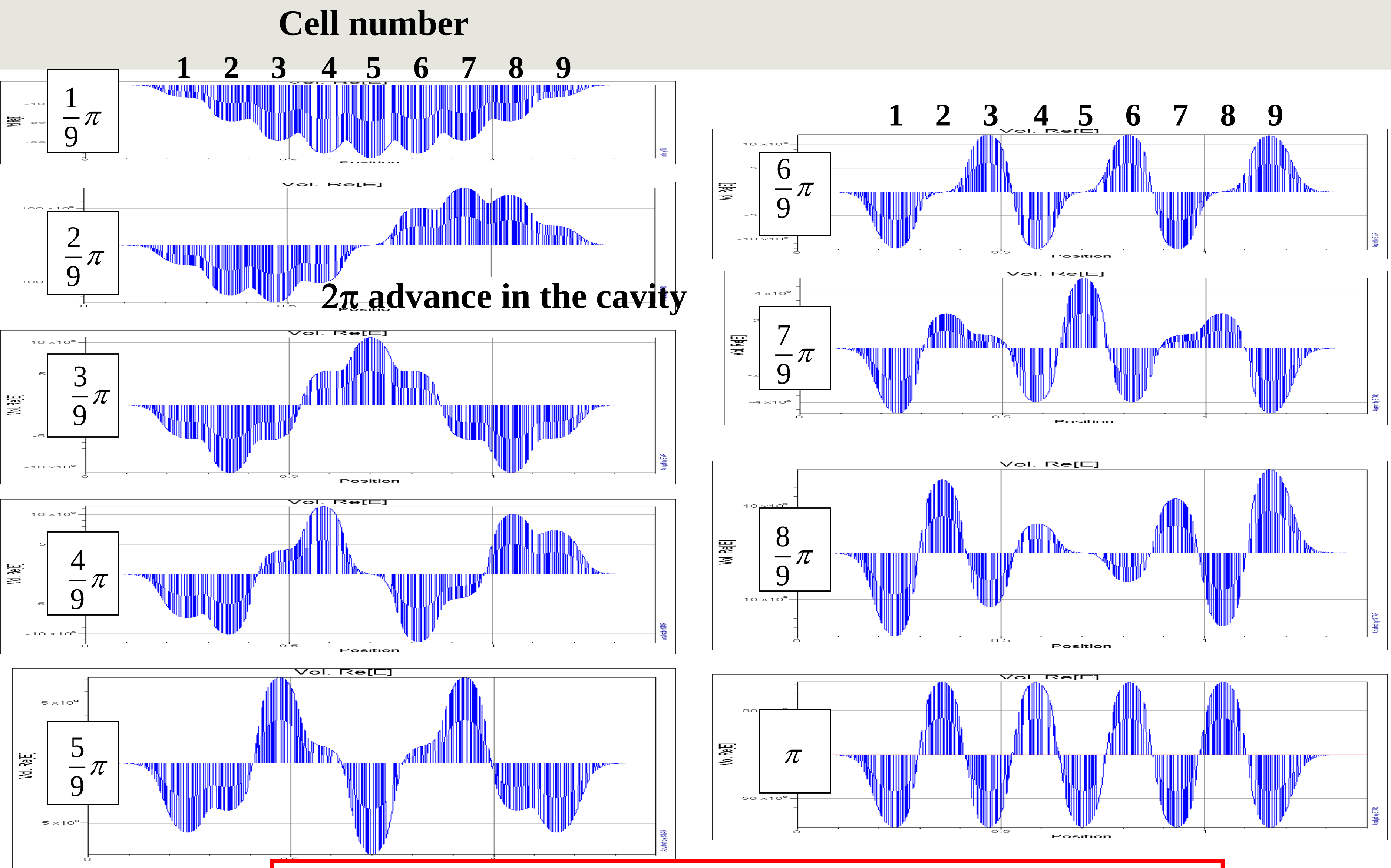
$$-1-4k = -\Omega^{(N)} \rightarrow \quad \gamma = 2k$$

$$v_j^{(m)} = B^{(m)} \sin\left[m\pi\left(\frac{2j-1}{2N}\right)\right], \quad B^{(m)} = \sqrt{\frac{2-\delta_{m,N}}{N}}$$

m = m-th mode,

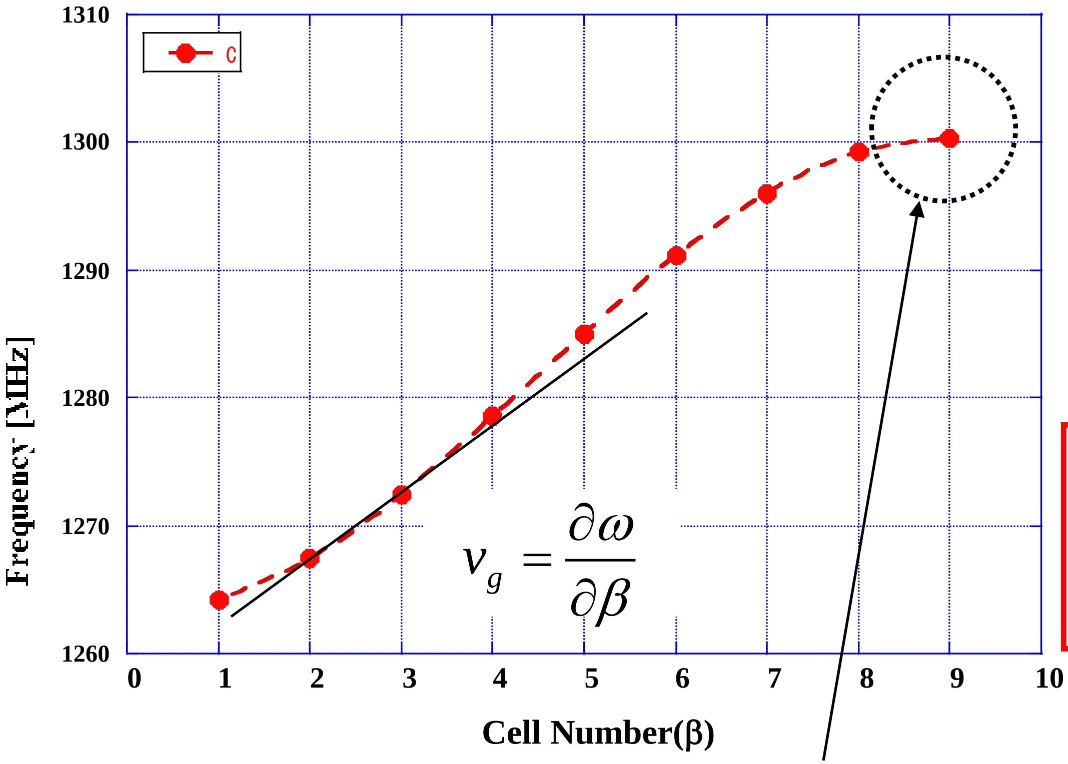
j = j-th cell in mode m-th

Pass-band modes of TM_{010} in a 9-cell cavity



Phase advance by mπ/9 per one cavity (9- cell).

Group velocity in the pass-band modes



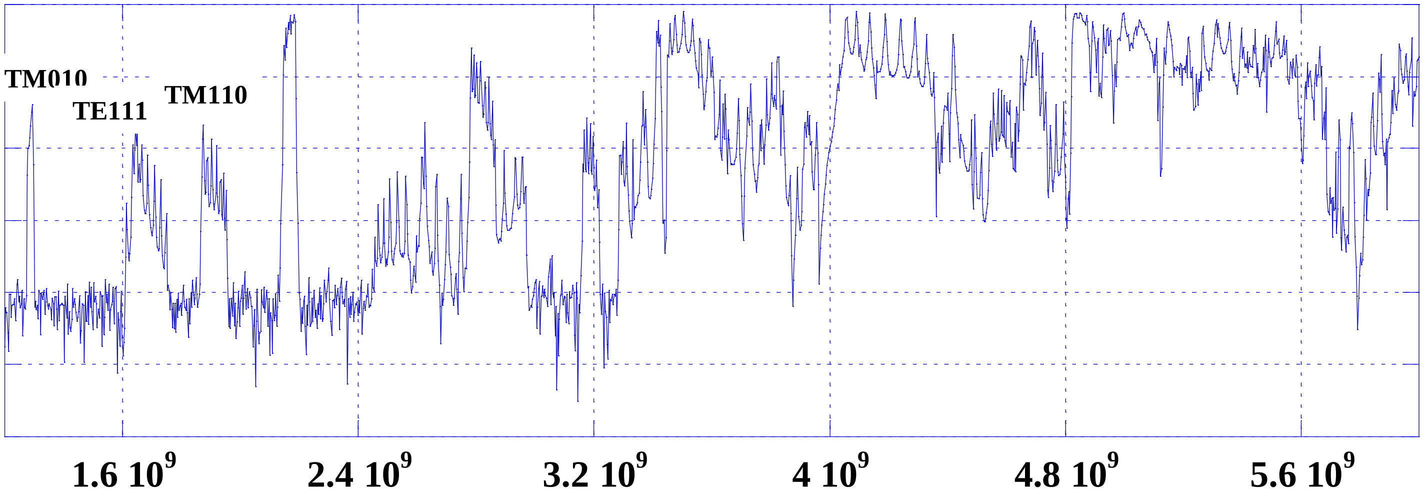
$$v_g = \frac{\partial \omega}{\partial \beta}$$

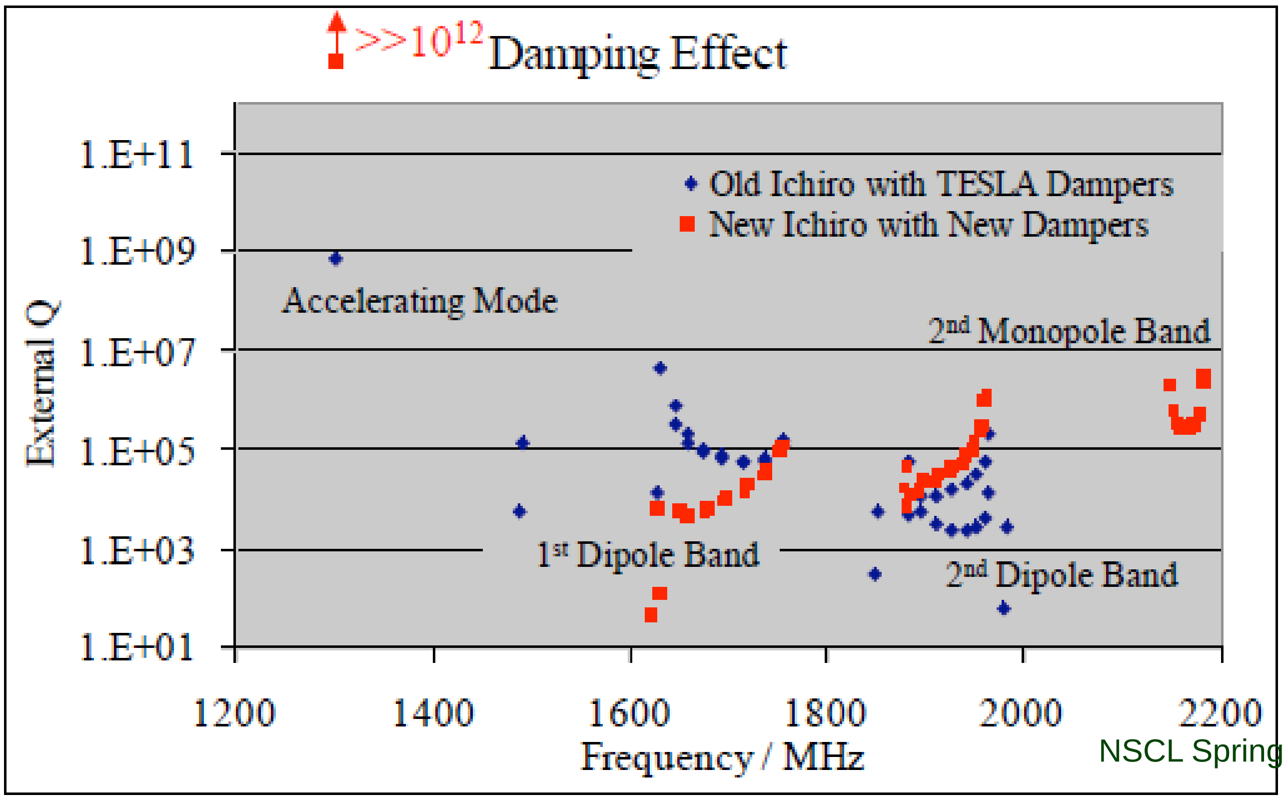
$$v_{j-1}^{(m)} + v_{j+1}^{(m)} = B^{(m)} \sin\left[\left(m\pi \frac{(2j-1)}{2N}\right) - \frac{m\pi}{N}\right] + B^{(m)} \sin\left[\left(m\pi \frac{(2j-1)}{2N}\right) + \frac{m\pi}{N}\right] = 2v_j^{(m)} \cos\left(\frac{m\pi}{N}\right)$$

$$\boxed{\Omega^{(m)} = \left(\frac{f_m}{f_0}\right)^2 = 1 + 2k\left[1 - \cos\left(\frac{m\pi}{N}\right)\right]}$$

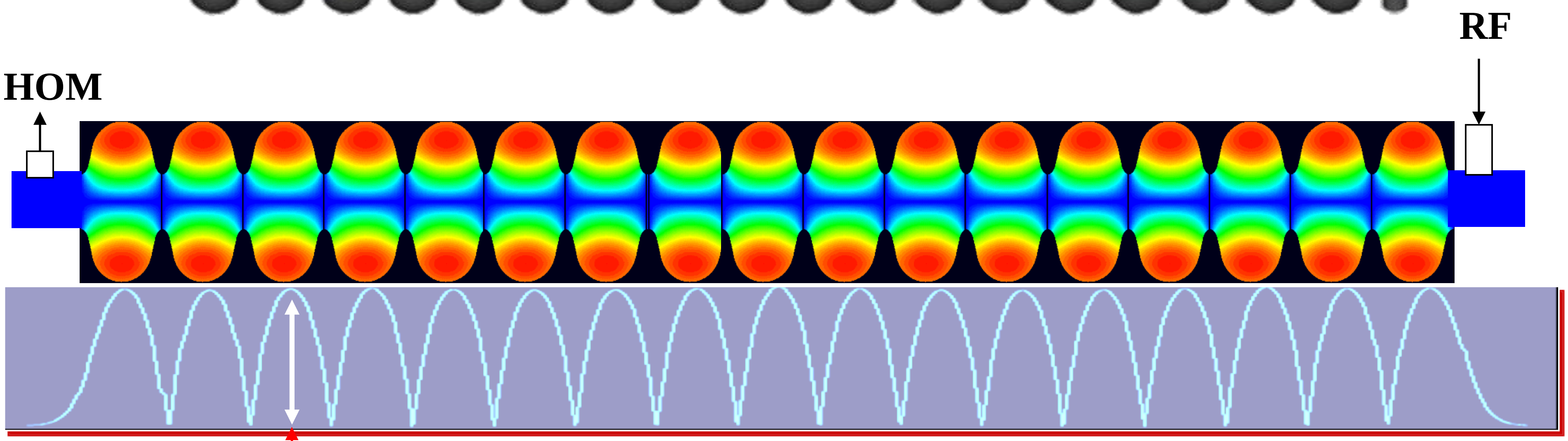
When the number of cells becomes infinity, v_g(π-mode)=0, which means no RF flow between nearby cells. Beam acceleration becomes unstable at large cell number. This is a demerit of the multi-cell structure.
Usually for very high current machine (KEKB), single cell cavity is chosen.

HOM modes in Ichiro 9-cell Cavity





How to decide the number of cells



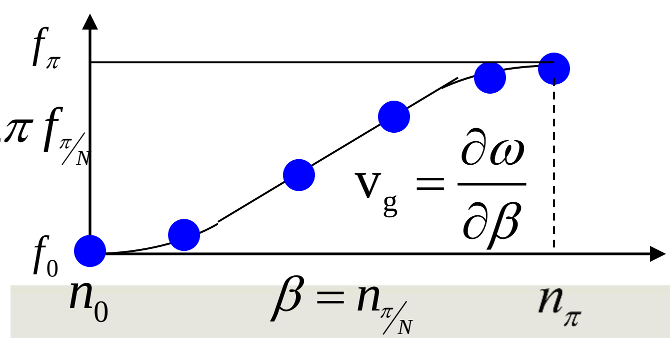
$$\frac{\Delta A_i}{A_i} = a_{ff}\,\frac{\Delta f_i}{f_i} = \frac{N^2}{k_{cc}} \cdot \frac{\Delta f_i}{f_i}$$

Beam pipe has no acceleration beam.
BP reduce the efficiency.
Multi-cell is more efficient.

N: number of cells

Field flatness factor : $a_{ff} = \dfrac{N^2}{k_{cc}}$

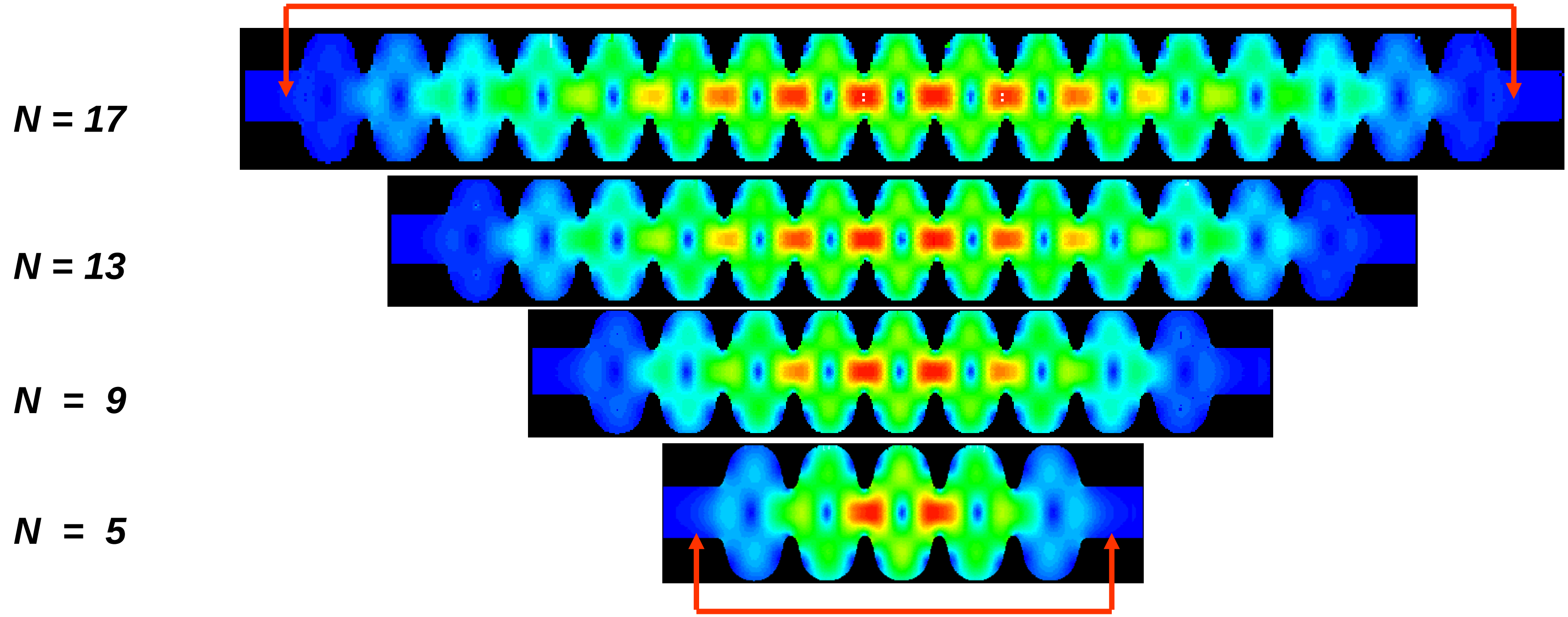
Cell to cell coupling : $k_{cc} = 2 \cdot \dfrac{f_\pi - f_0}{f_\pi + f_0}$



Large number is better for stable beam operation

HOM trapping vs. number of cell

No fields at HOM couplers positions, which are always placed at end beam tubes



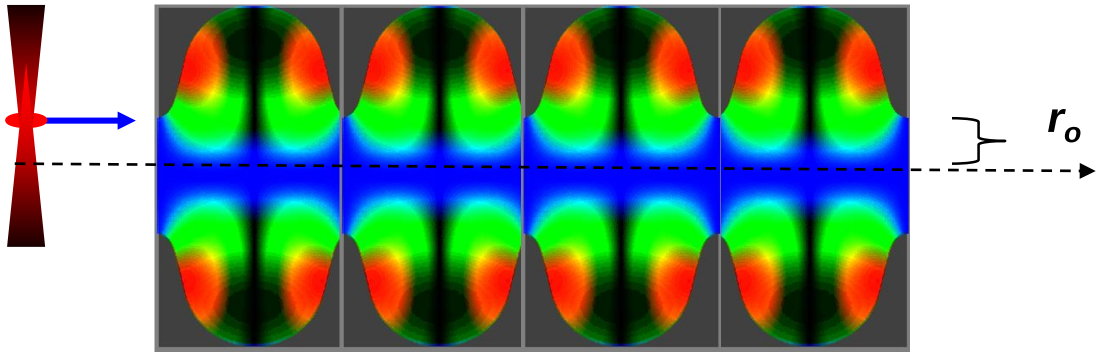
e-m fields at HOM couplers positions

Smaller number of cells is easy to take out HOMs.

HOM (Higher Order Mode)

The beam will excite HOM modes, if it passes off beam axis of the cavity.

J.Sekutwitz' s Slide



*The amount of induced energy by charge **q** is:*

$$\Delta U_q = k_{\parallel} \cdot q^2 \quad \text{for monopole modes (max. on axis)}$$

$$\Delta U_q = k_{\perp} \cdot q^2 \quad \text{for non monopole modes (off axis)}$$

where $k_{\parallel}$ *and* $k_{\perp}(r)$ *are loss factors for the monopole and transverse modes respectively.*

The induced E-H field is a superposition of <u>cavity eigenmodes</u> having the component of the <u>electric field along the trajectory</u>.

HOM problem

J.Sekutwitz' s Slide

Two kind of phenomena can limit performance of a machine due to the beam induced HOM power:

- *Beam Instabilities and/or dilution of emittance*
- *Additional cryogenic power and/or overheating of HOM couplers output lines*

<u>Beam instabilities and/or dilution of emittance</u>

Transverse modes (dipoles) causing emittance growth+ monopoles causing energy spread
This is the main problem
***in linacs**: TESLA or ILC, CEBAF, European XFEL, linacs driving FELs.*

<u>Additional cryogenic power and/or overheating of HOM couplers output lines</u>

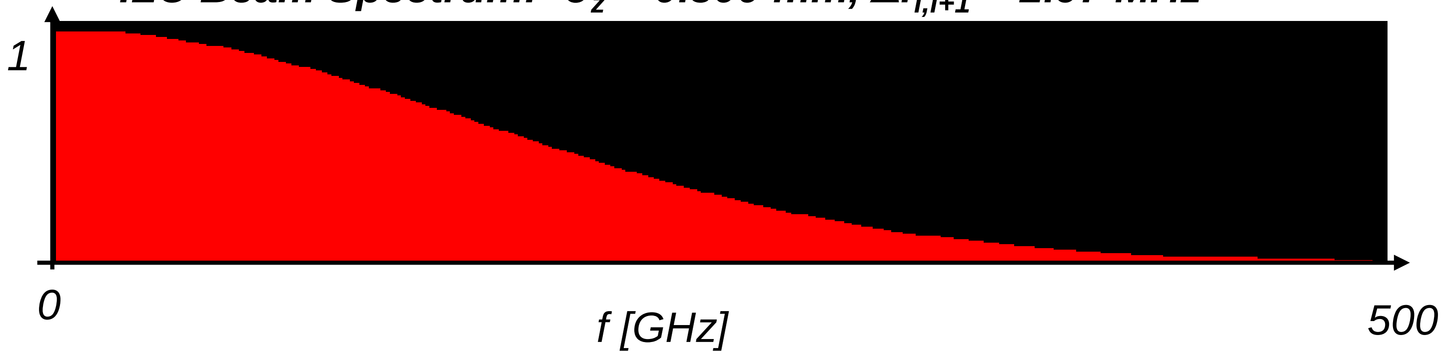
Monopoles having high impedance on axis are excited by the beam and store energy which must be coupled out of cavities, since it causes additional cryogenic load, and induces energy spread.
This is the main problem
***in high beam current** machines: B-Factories, Synchrotrons, Electron cooling.*

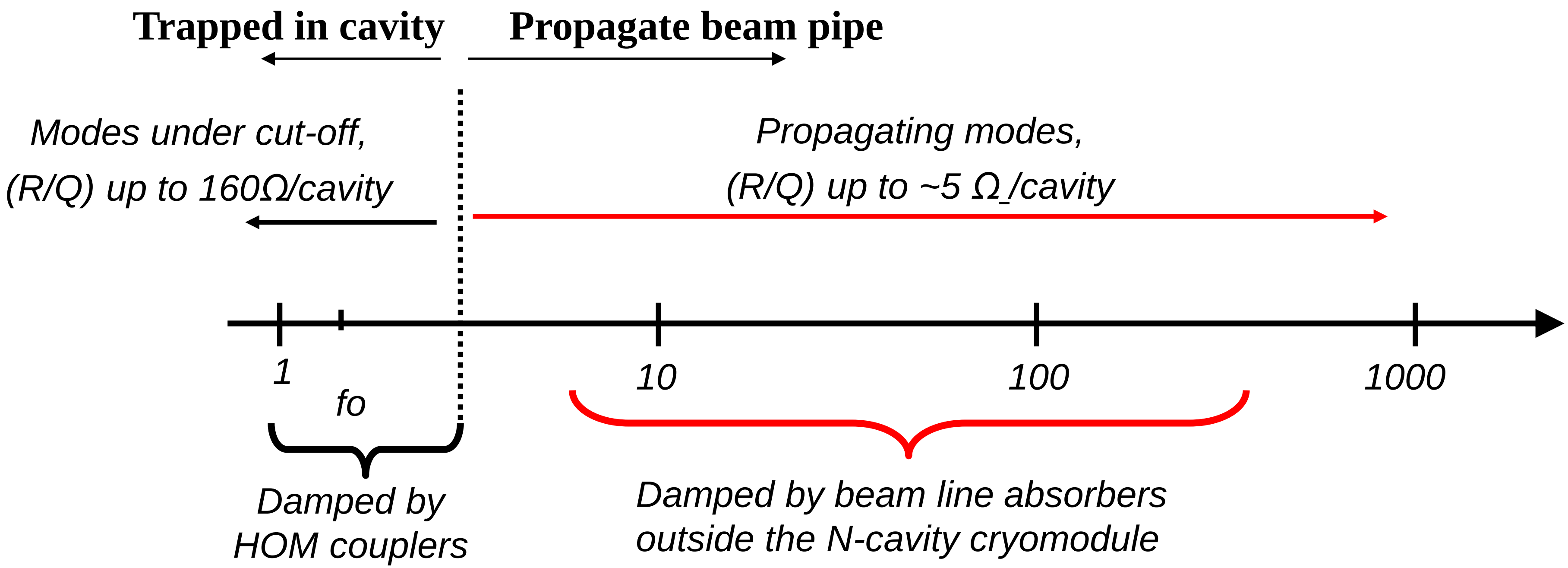
HOM modes have to be taken out from the cavity through HOM coupler.

Propagating Modes and Trapping between Cavities

Spectra of accelerated beams, which bunches are shorter than 1 mm, extend to hundreds of GHz.

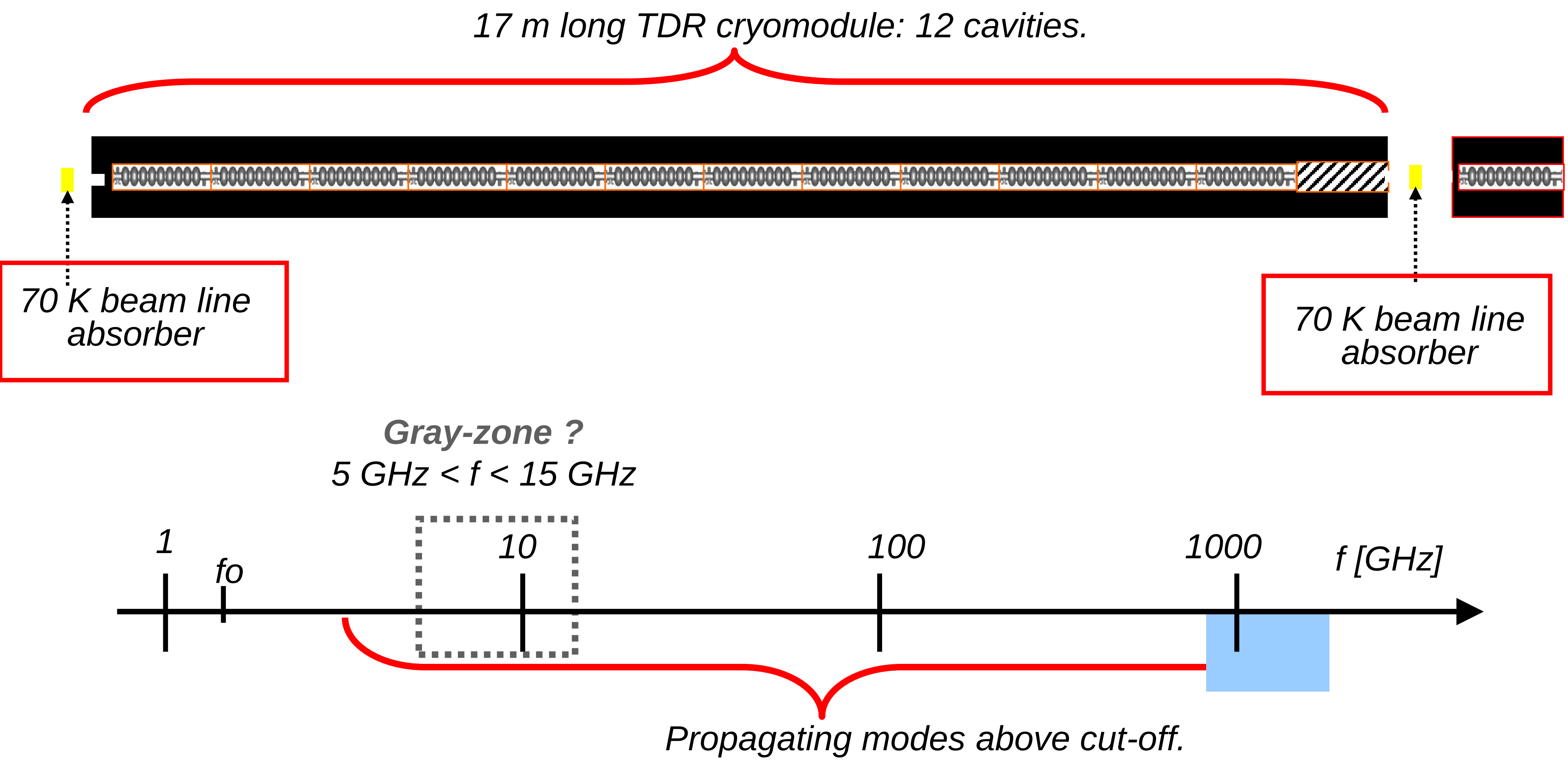
ILC Beam Spectrum: σ_z = 0.300 mm, $\Delta f_{i,i+1}$ = 2.97 MHz





Damping the HOM with higher frequency over than cut-off frequency

Propagating Modes and Trapping between Cavities ***HOM***

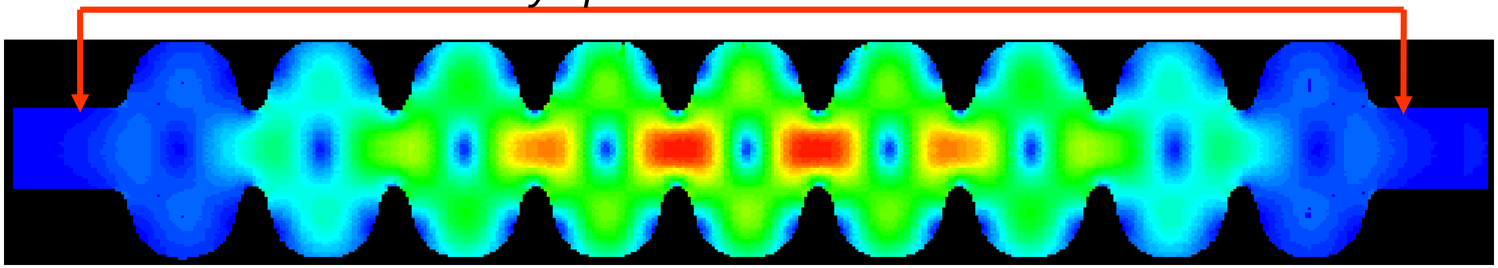


Attention to trapped modes damping through HOM coupler

Trapping of Modes within Cavities ***HOM***

HOM couplers limit RF-performance of sc cavities when they are placed on cells

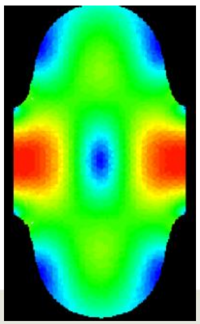
no E-H fields at HOM couplers positions, which
are always placed at end beam tubes



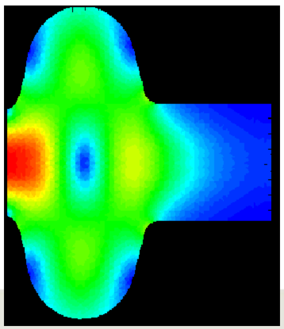
The HOM trapping mechanism is similar to the FM field profile unflatness mechanism:

- *weak coupling HOM cell-to-cell,* $\boldsymbol{k}_{cc,HOM}$
- *difference in HOM frequency of end-cell and inner-cell*

f = 2385 MHz



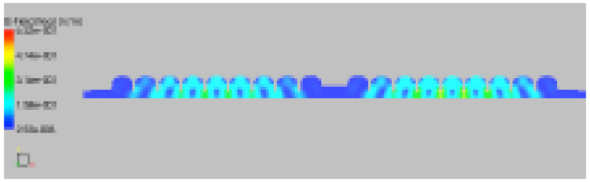
That is why they hardly resonate together



f = 2415 MHz

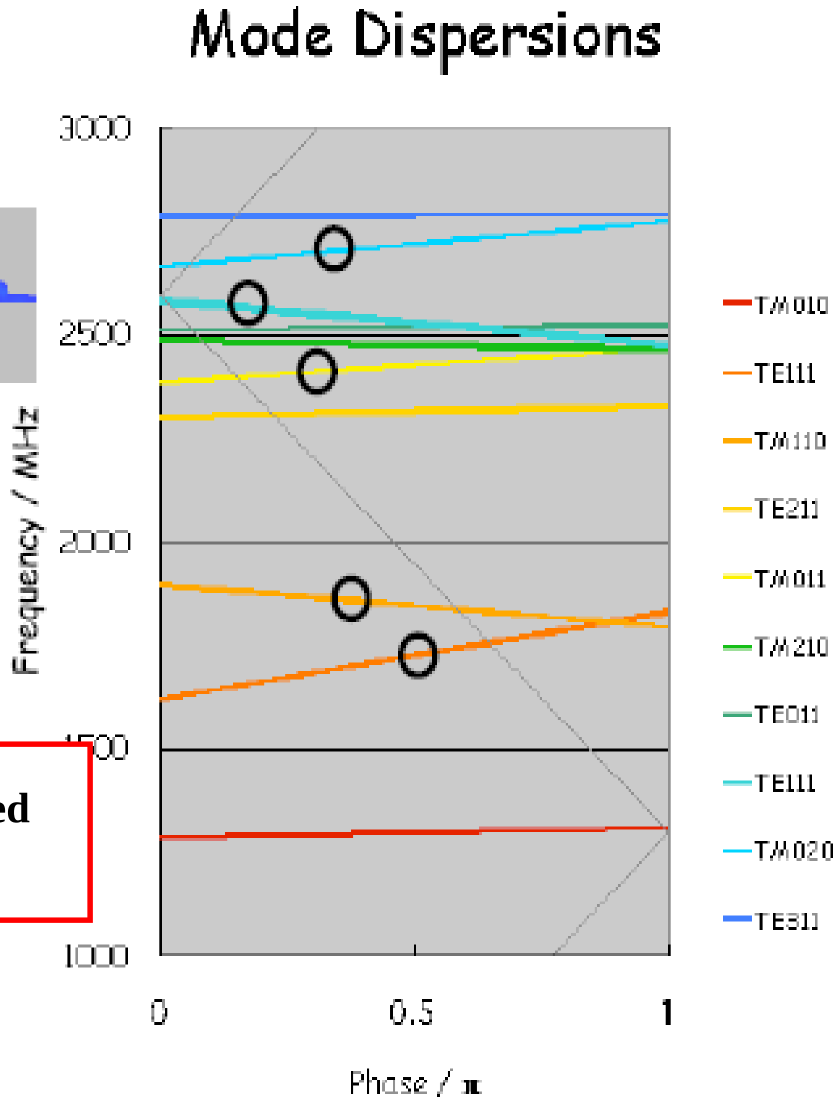
Mode Dispersions in Ichiro Cavity

Trapped Dipole Mode ?



17 V/(pCm^2)

Trapped modes can be formed at the mode crossing.



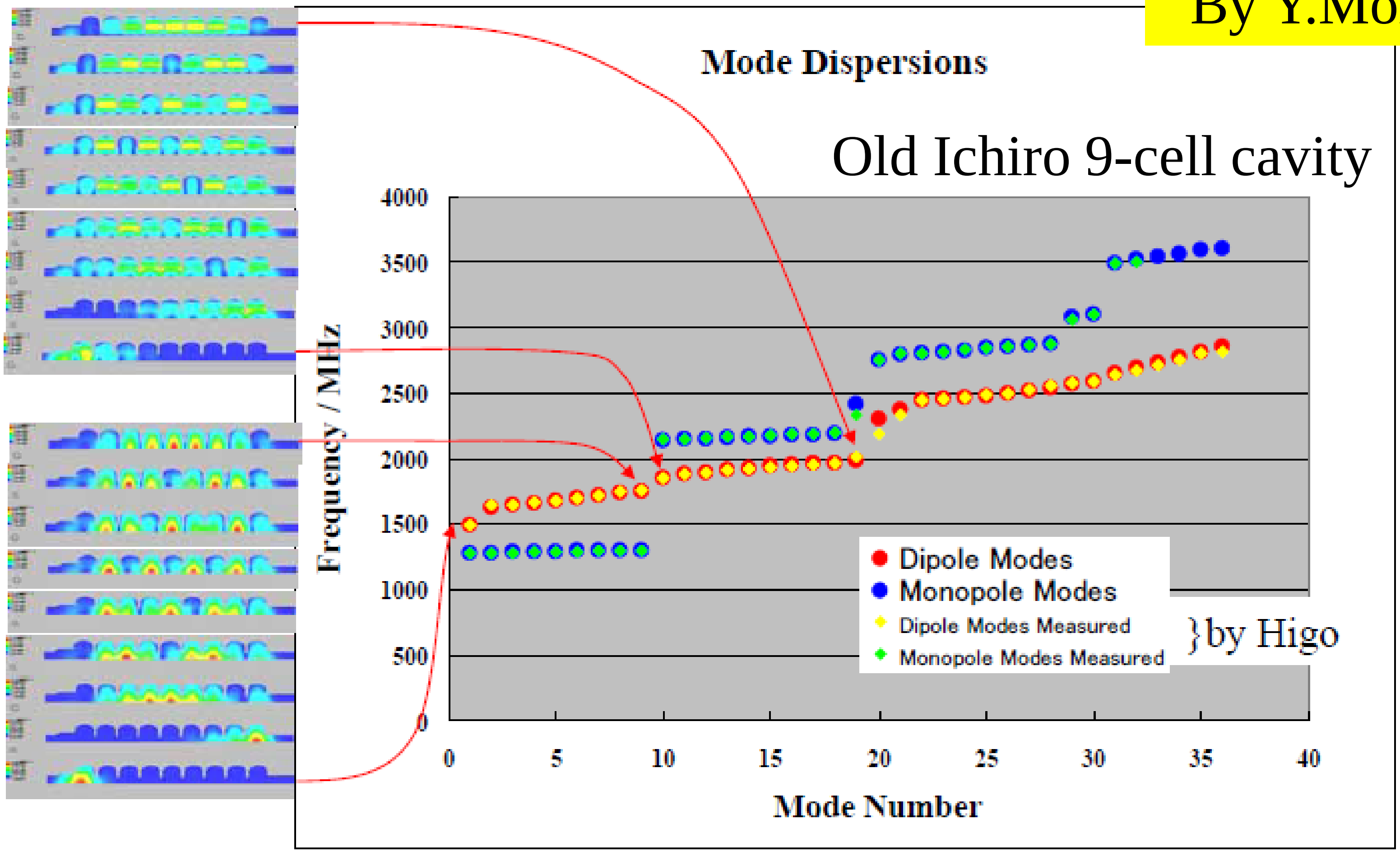
Monopole Modes
~ 3500 MHz
Dipole Modes
~ 2500 MHz

Surveyed up to now

Mode Dispersions

Simulated for 1st 9-Cell Structure without Coupler Ports
(Beam Ports Terminated)

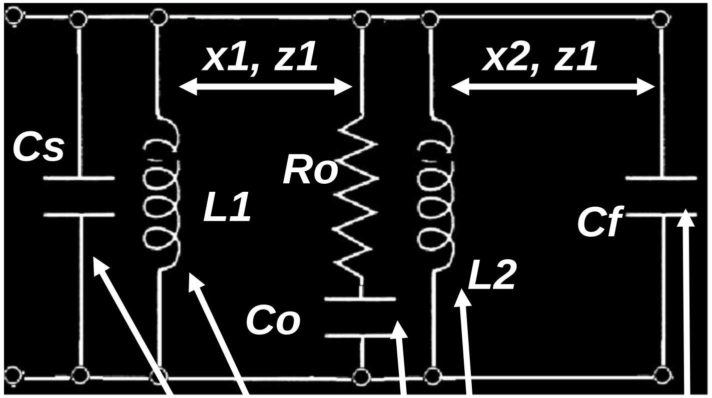
By Y.Morozumi

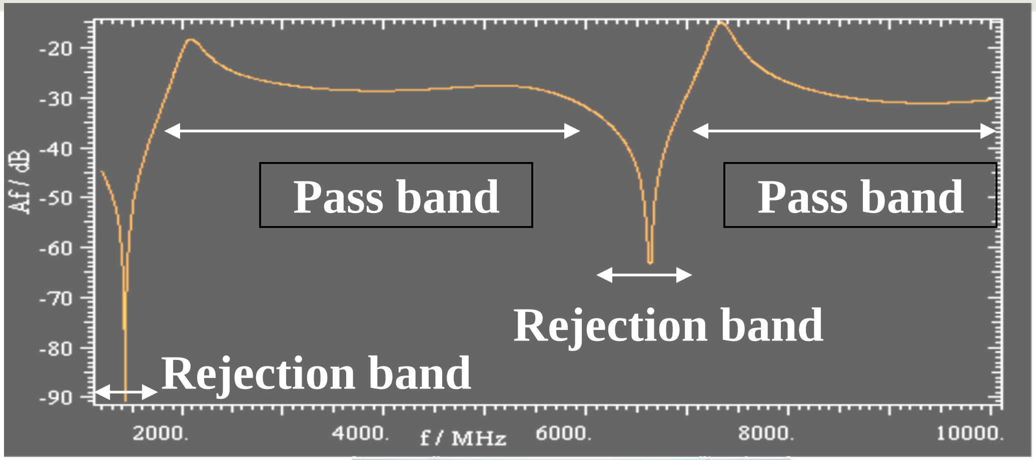


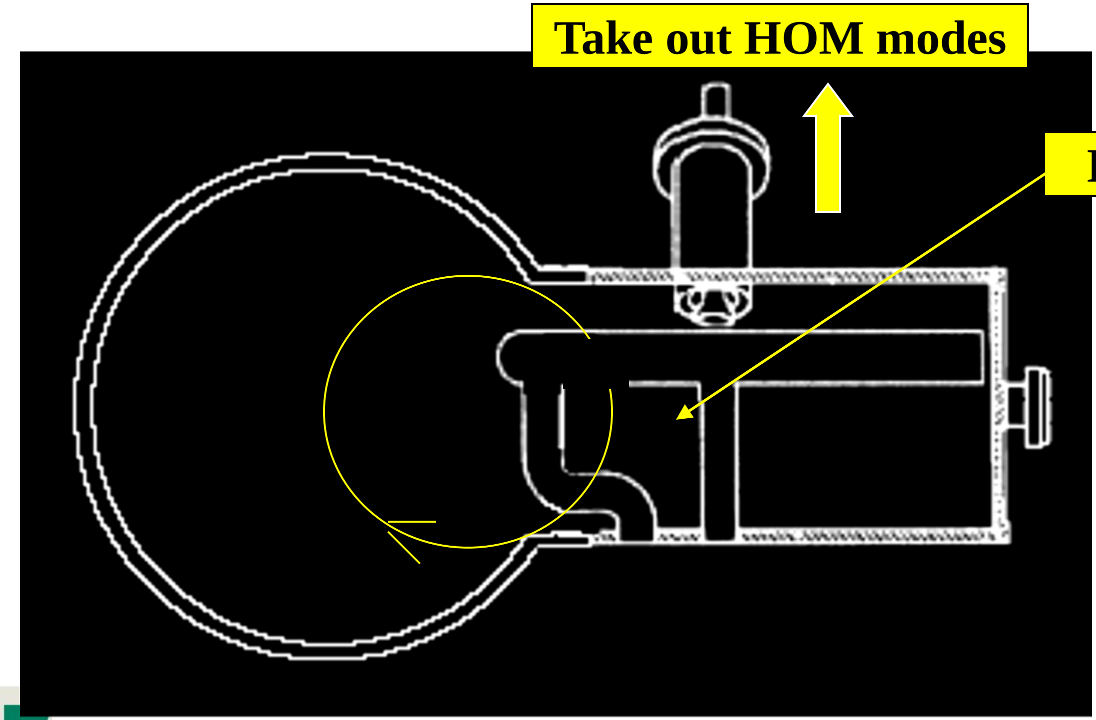
11-13/May/2011K.Saito

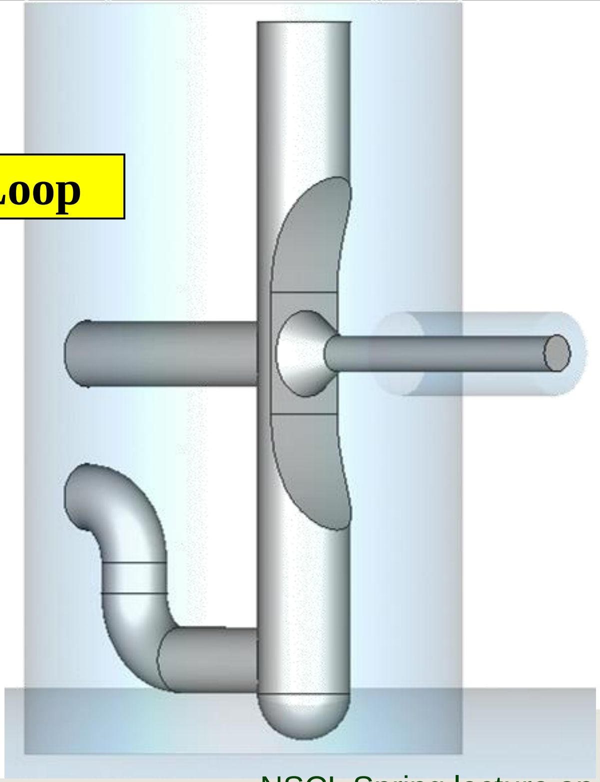
HOM coupler

The TESLA –like HOM couplers are nowadays designed in frequency range: 0.8-3.9 GHz

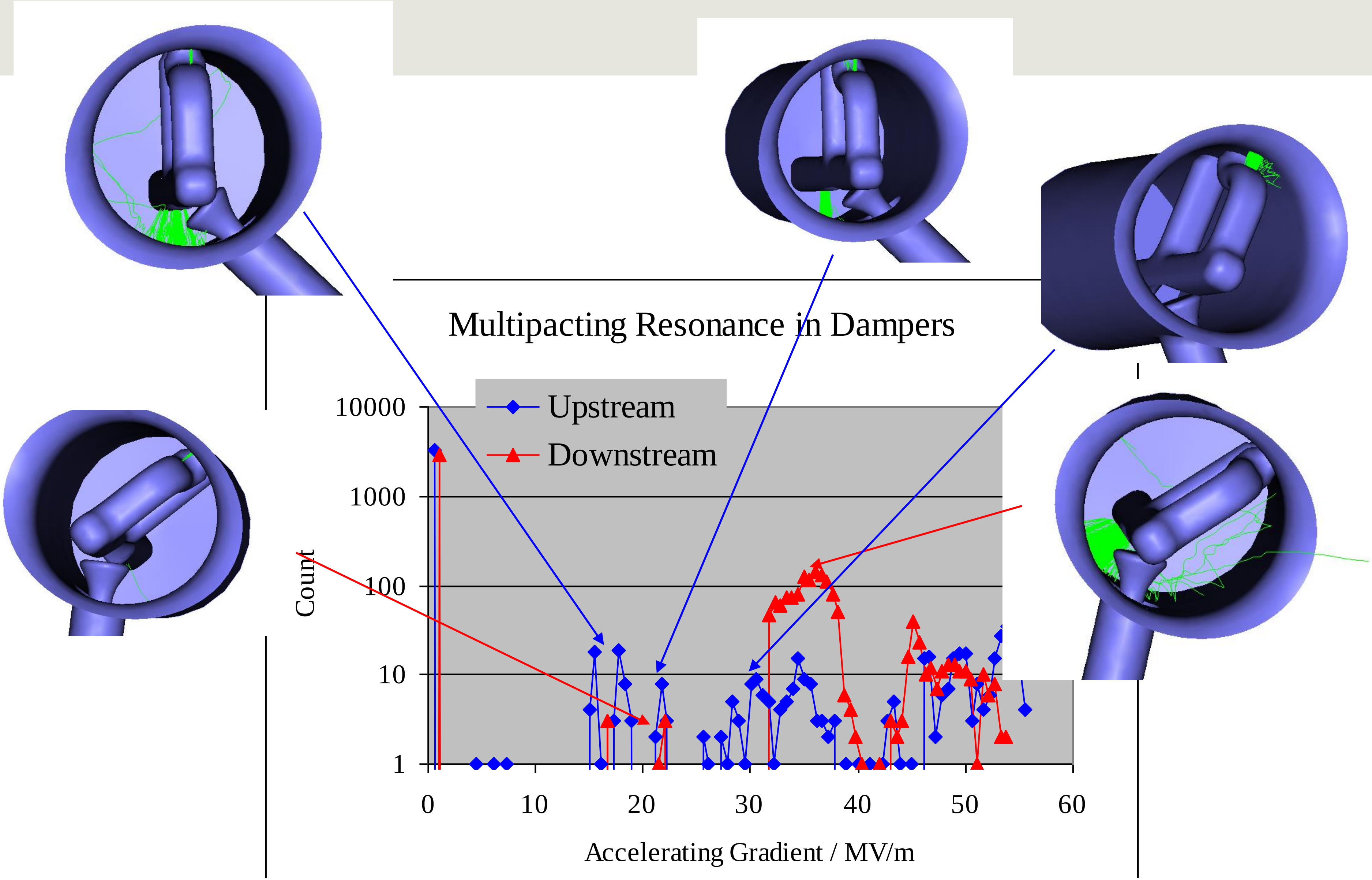






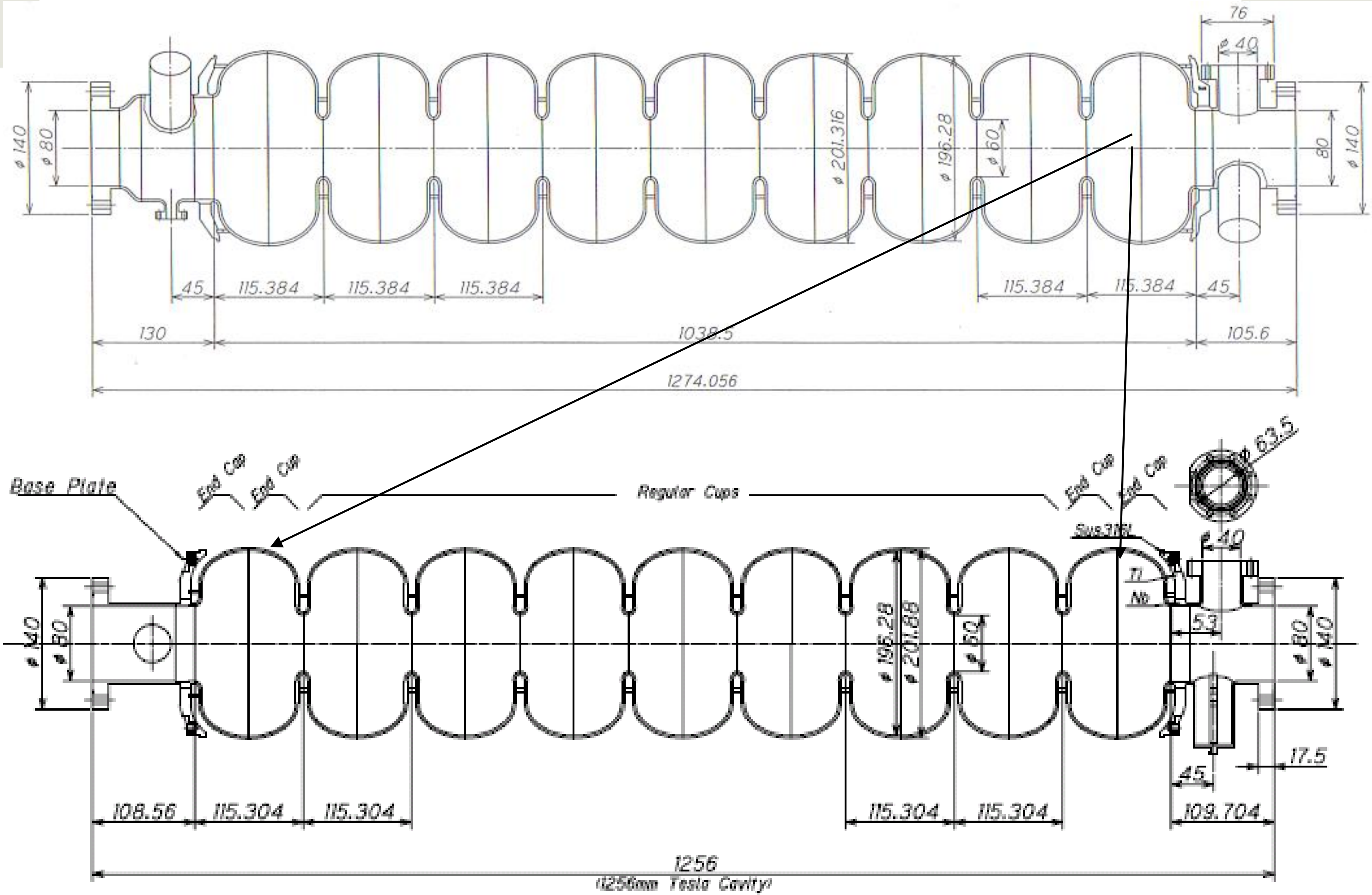


Multipacting in current HOM dampers



Might be processed out easily, Under redesigning for new Ichiro cavity

Re-designed Cavity Shape: Old Ichiro/New Ichiro



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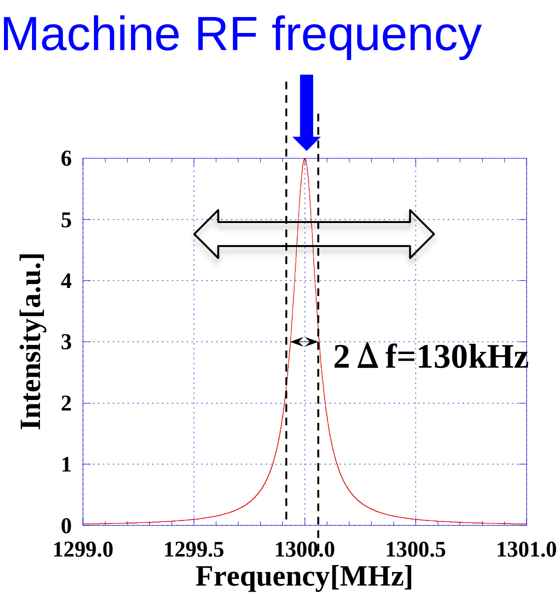
Cavity design comparison with TESLA shape

	TTF	STF 45MV/m	Advantage/Disadvantage
Operation gradient	23.4MV/m 35MV/m XFEL TESLA800	45MV/m	1-1.1 TeV in 40km tunnel. If superstructure, 33km for 1TeV
Cavity shape	Iris ϕ70 Beam pipe ϕ78 cell taper 14^{O}	Iris ϕ60 Beam pipe ϕ80/ϕ108 cell taper 0^{O}	Tighter tolerance due to severe wake field
R/Q[Ω]	1036	1144	10% higher electric efficiency
Ep/Eacc	2.00	2.31	
Hp/Eacc[Oe/(MV/m)]	42.6	37.8	Eacc max 46-49MV/m
Cell-to-cell coupling	1.86	1.55	F-flatness more sensitive on fabrication error
Tolerance[μm]	250	170	
Sensitivity for Lorenz detuning[$Hz/(MV/m)^2$]	1	1.18	Wide detuning range
Lorenz detuning[Hz]	200 600	~1500	Need wide range tuner
He jacket material	5t Ti	3t SUS316L	Japanese high pressure code has no concern of these materials
RF transmission power[kW]	240 350	500	Need higher input coupler

7.10 SRF cavity frequency control in machine operation

- Cavity frequency has to be kept the same as the machine RF frequency within the half bandwidth (fixed frequency, for instance 322.000MH_Z for FRIB HWRs) in machine operation
- SRF cavity has a very high Q and the very narrow bandwidth even consider Q_L, for instance FRIB HWR 30Hz against 322MHz ($1/10^7$).
- SRF cavity easily changes the cavity frequency by noises:
  - LHe pressure variation (very small in 2K operation)
  - Lorentz detuning (depends on gradient, not so much changes during CW operation)
  - Mechanical vibration
- SRF frequency has to be fixed by tuners
  - Mechanical tuner (slow)
  - Piezo tuner (fast)
- Microphonic excitation by environment noises through the cavity mechanical modes has to be reduced.
  - When the noise frequency match the cavity mechanical mode frequency, the cavity frequency change is resonantly enhanced
- Mechanical design is important.

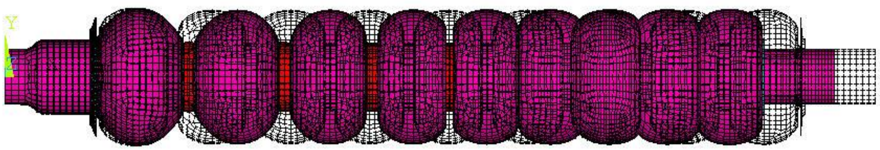


7.11 Mechanical design

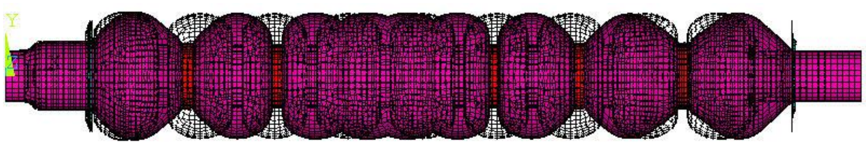
Naked Cavity

69Hz

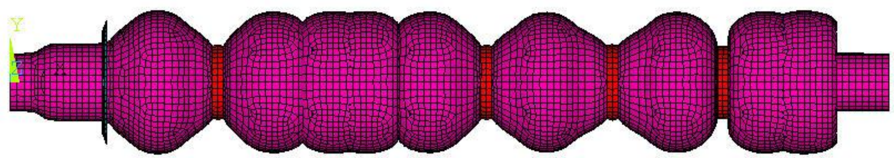
207Hz



343Hz

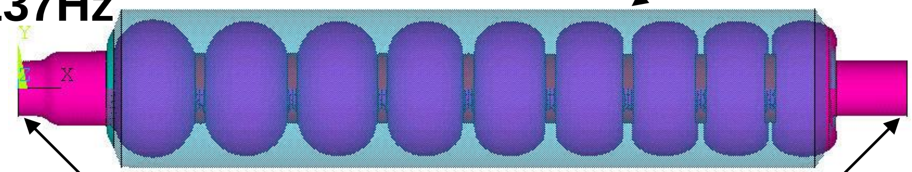


476Hz

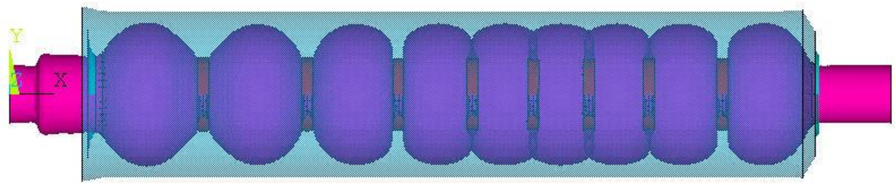


With He Vessel

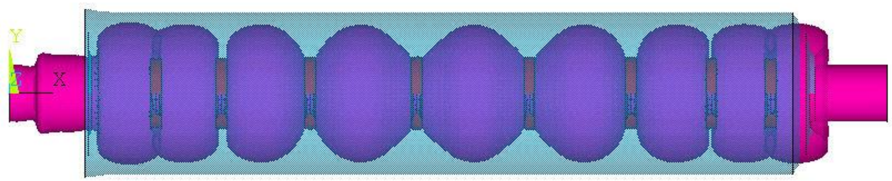
137Hz



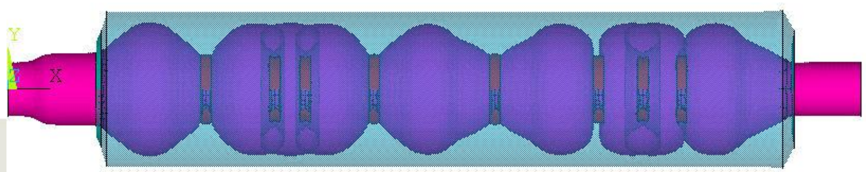
260Hz



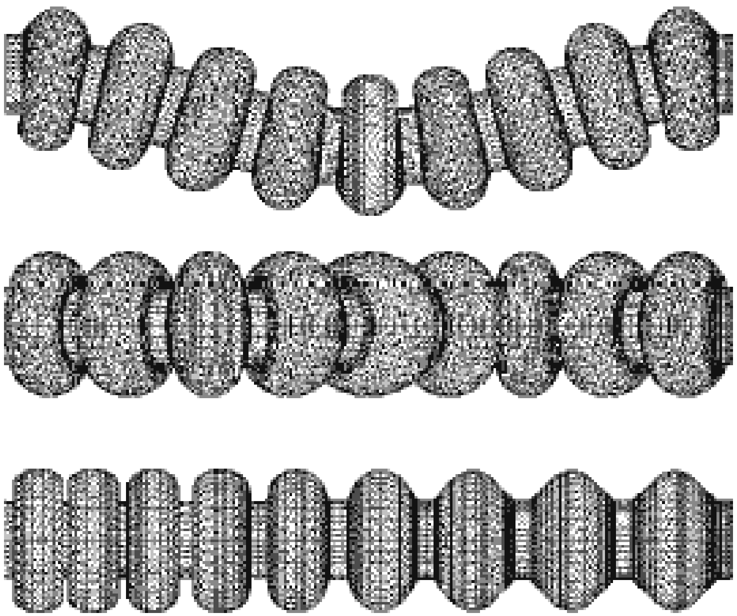
313Hz



569Hz



Mechanical Resonance of a multi-cell cavity



60 Hz

152 Hz

Transverse modes

250 Hz *Longitudinal mode*

TESLA structure

The mechanical resonances modulate frequency of the accelerating mode.
Sources of their excitation: vacuum pumps, ground vibrations…

Mechanical resonance excitation

$$m\frac{d^2x}{dt^2}+\lambda\frac{dx}{dt}+kx=mf\sin(\omega t):\quad \text{Forced oscilation}(\omega)$$

$$x=ae^{-\mu t}\sin(\omega_1 t+\phi_1)+A\sin(\omega t-\varphi)$$

$$\omega_0=\sqrt{\frac{k}{m}},\ \omega_1=\sqrt{\frac{k}{m}-\frac{\lambda^2}{4m^2}},\ \mu=\frac{\lambda}{2m},$$

$$A=\frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+4\mu^2\omega^2}},\quad \tan(\varphi)=\frac{2\mu\omega}{\omega_0^2-\omega^2}$$

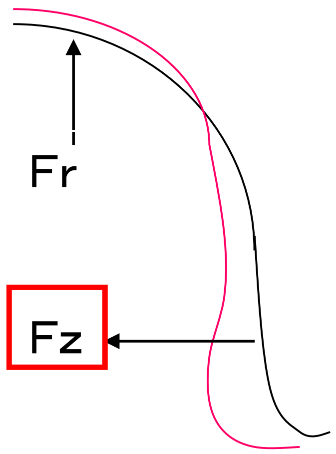
Mechanical vibration amplitude becomes maximum at $\omega=\omega_r=\sqrt{\omega_0^2-2\mu^2}\approx\omega_0$

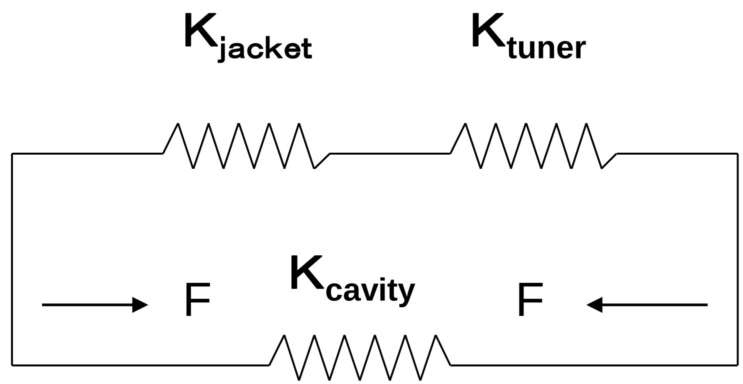
$$\mu^2<<\omega_0^2\ ,\omega_0\approx\omega_1\approx\omega_r$$

$$A^2(\omega)\approx\frac{A_r^2}{(\omega-\omega_r)^2+\mu^2},A_r=\frac{f}{2\mu\omega},2\mu\approx\omega_r-\omega_-=\Delta,\ \boxed{Q_m=\frac{\omega_r}{\Delta}}$$

Tow components of Lorentz deformation

Noguchi's slide in the 1st ILC school





$$\Delta f = \sum_{\text{mode}} a_k \, \delta f_k \approx \sum_{\Delta l = 0} a_k \, \delta f_k + \frac{df}{dl}\frac{dl}{dF} F$$

$$= A E_{acc}^2 + \boxed{\frac{df}{dl}\frac{B E_{acc}^2}{K_S}}$$

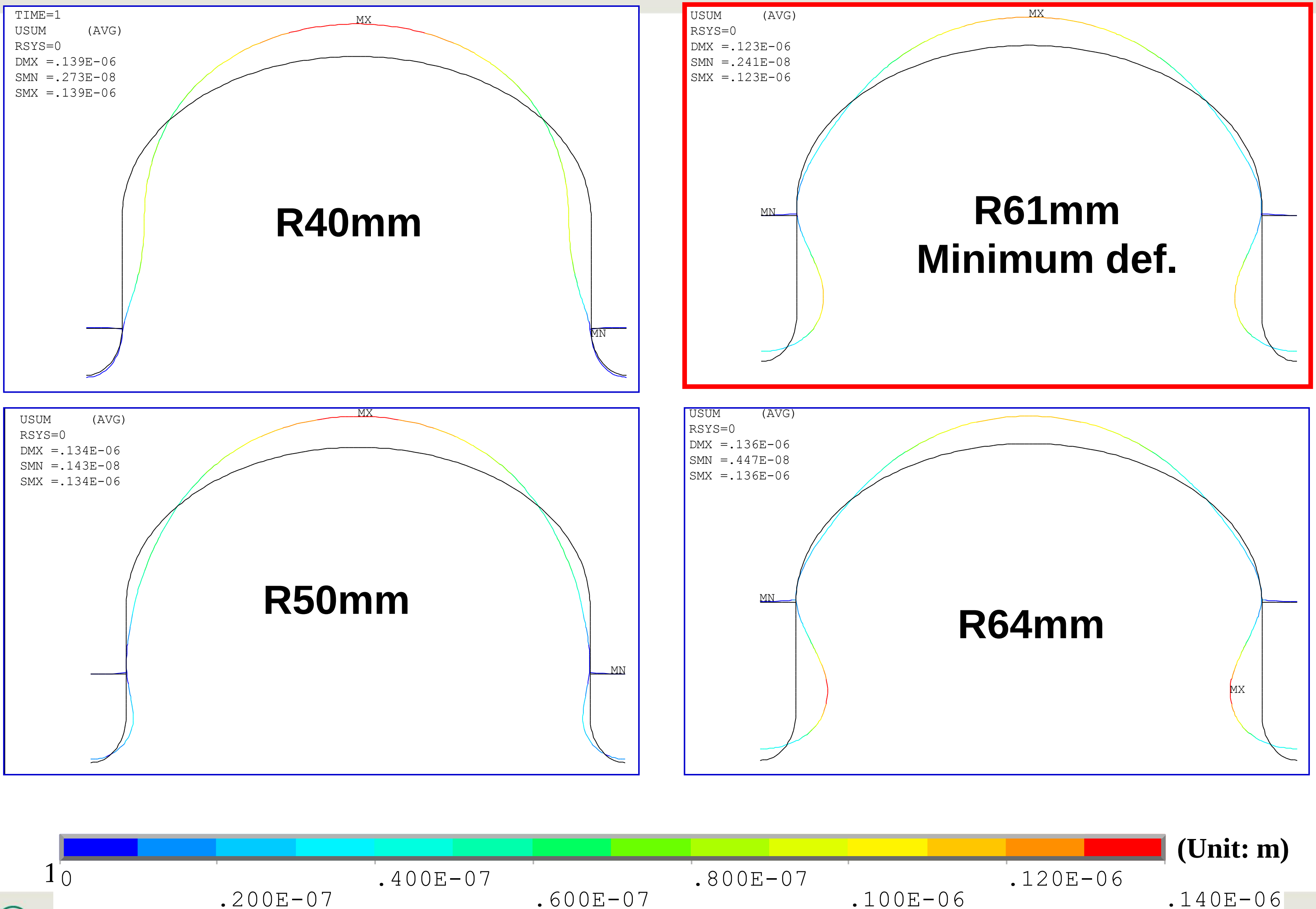
Rigid Stiffness at Jacket and Tuner are also Very important against the Lorentz Detuning.

		TESLA Blade	STF Slide Jack	STF Ball Screw
A	Hz/(MeV/m)^2	0.5	0.5	(1.2)
B	N/(MeV/m)^2	0.047	0.047	0.051
df/dl	Hz/μm	320	320	370
dF/dl	N/μm	3	3	1.8
KS	N/μm	13	80	60
Kjacket	N/μm	26	96	58
Ktuner	N/μm	26	500	1700
Δf (30MV/m)	Hz	1490	620	(1360)
Fine Tuning Stroke	μm	3.7	1	2.9

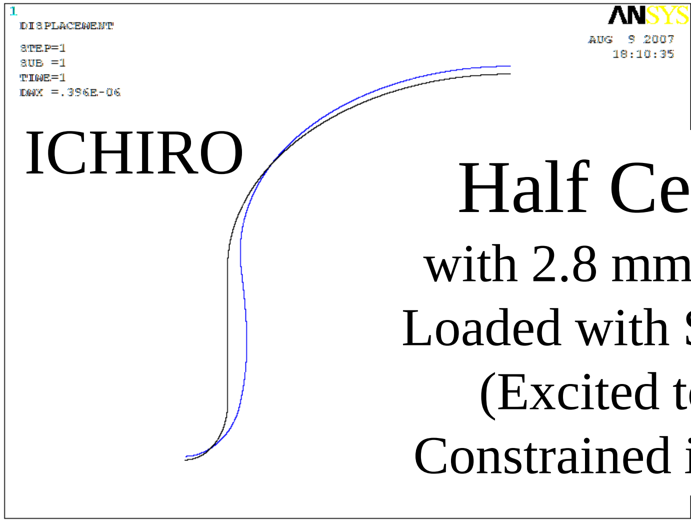
Optimization of Stiffener Location against Lorentz Detuning

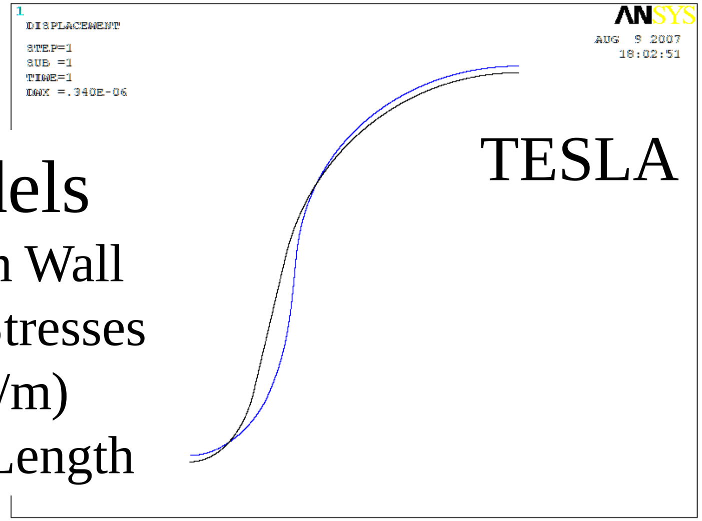
Eacc=38MV/m

by H.Yamaoka



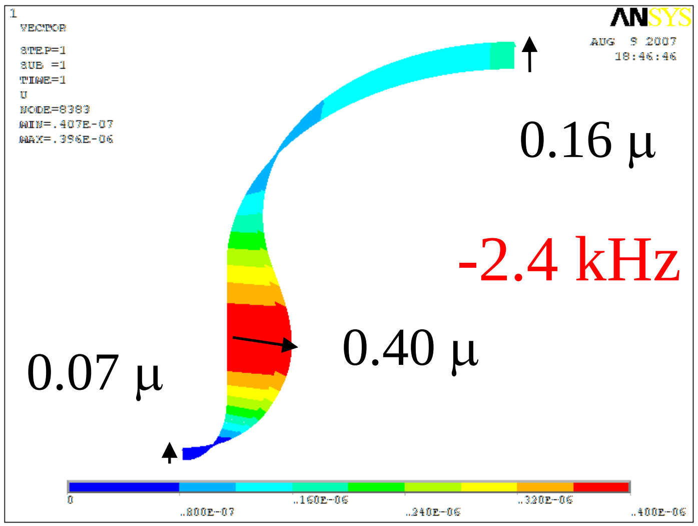
Comparison of Lorentz Force Deformation between different cell shapes

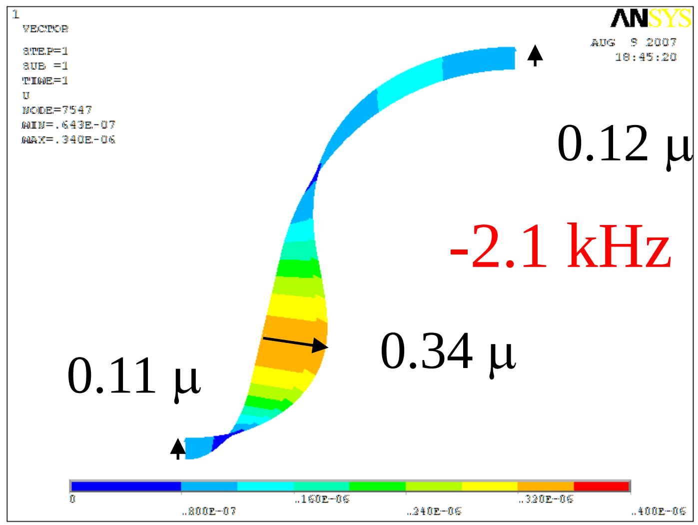




Half Cell Models

with 2.8 mm Niobium Wall
Loaded with Surface Stresses
(Excited to 40 MV/m)
Constrained in Axial Length





Backup Slides

Mode Assignment

TM_{mnp} mode :Hz=0,Ez≠0 cases

$$E_z = E_0 J_m\left(\frac{\rho_{mn}}{a} r\right)\cos(m\theta)\cos(\frac{p\pi z}{d})$$

m : number of nodes in θ direction
n : number of nodes in r direction
p : number of node in z direction

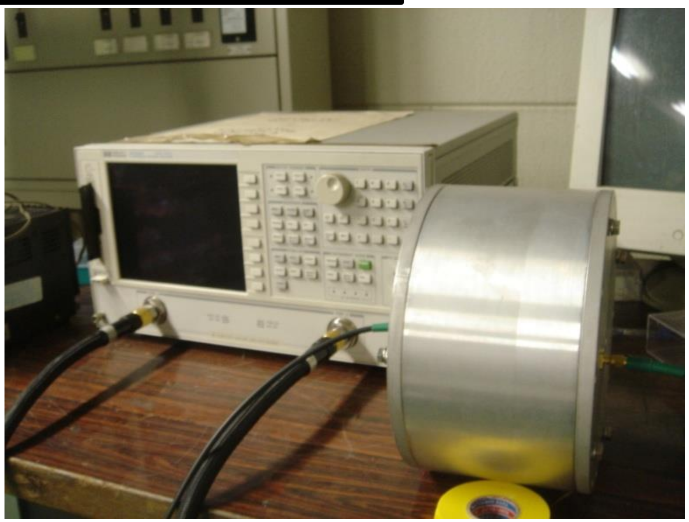
TE_{mnp} mode:Hz≠0,Ez=0 case

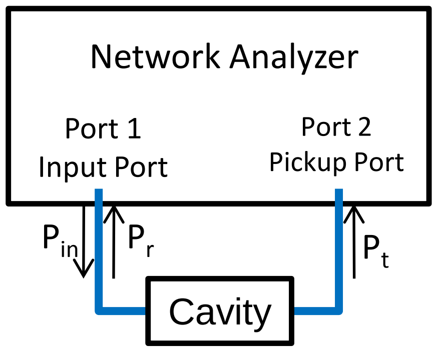
$$H_z = E_0 J_m\left(\frac{\rho'_{mn}}{a} r\right)\cos(m\theta)\sin(\frac{p\pi z}{d})$$

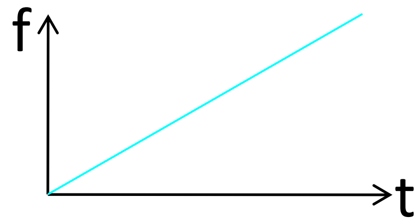
m : number of bellies in θ direction
n : number of bellies in r direction
p : number of bellies in z direction

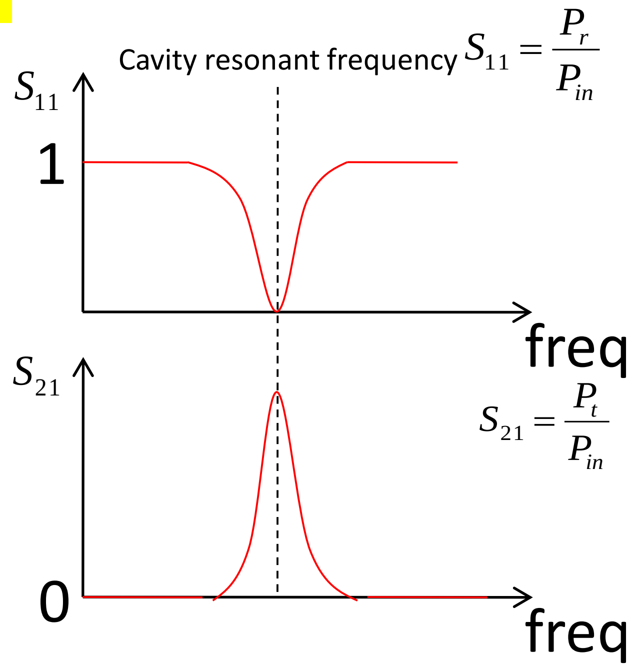
Measure the distribution along the axis

Network Analyzer







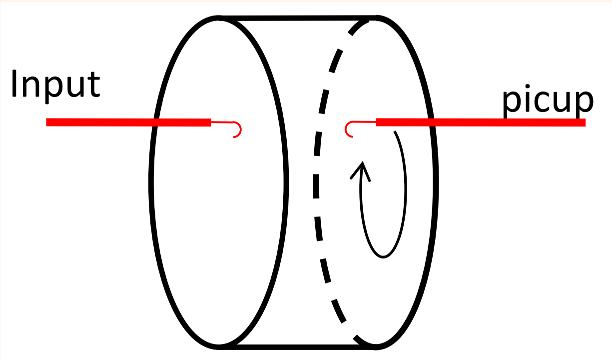


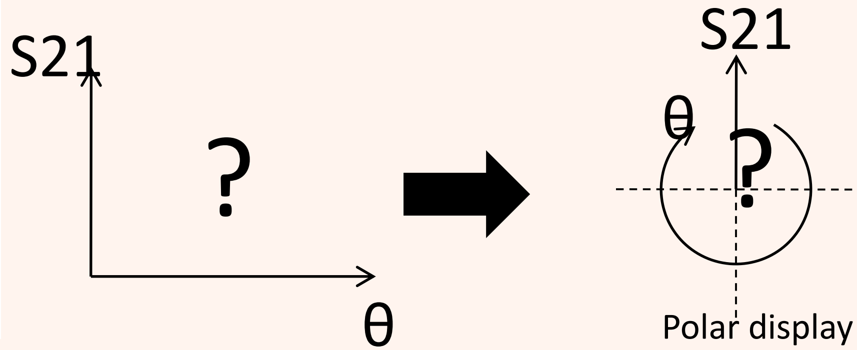
Network Analyzer: sweep frequency under the constant input power (Pin). Reflection (Pr)/transmitted (Pt) powers and the phase can be measured.

Let's S-parameters. Excite cavity through antenna.

Mode assignment continued

Distribution in Θ-direction : Fix the Input antenna,
Rotate the pick up antenna in θ direction looking S21

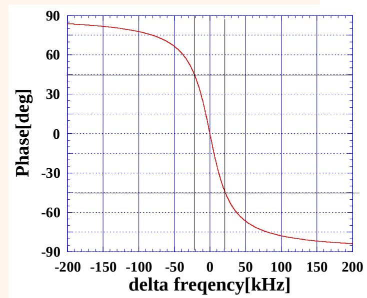


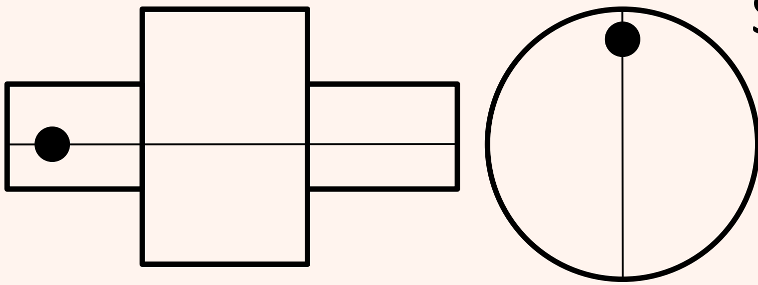


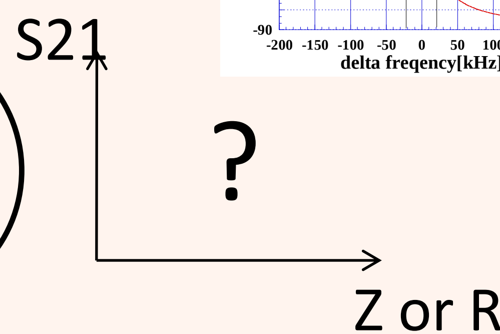
Distribution in R・Z:
Bead Pull Method

①Fix Input・Pickup
②Move a small metal piece on the axis at constant speed
③EM field can not penetrate in the metal piece and disturb the frequency.
④RF phase is varied at the fix frequency

$$\frac{\Delta U}{U_0} = \frac{\Delta f}{f_0} \qquad \Delta f = \frac{f_0}{2Q}\tan(-\Delta\varphi) \approx -\frac{f_0}{2Q}\Delta\varphi$$





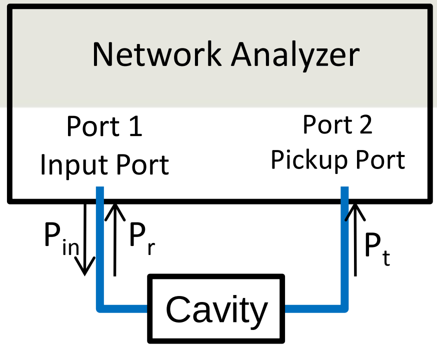


Q measurement method using Network Analyzer

Q_L (loaded Q) : Q value total loss included Cavity and Cable

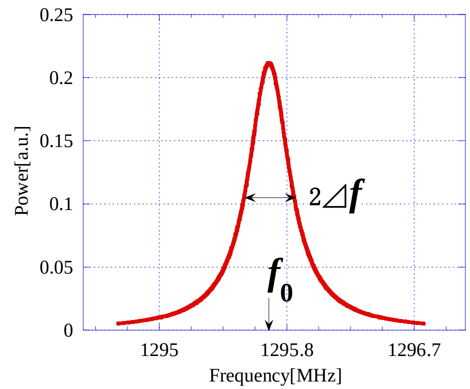
$$Q_L = \frac{\omega U}{P_{loss} + P_t + P_e} = \frac{f_0}{2\Delta f}$$

Wall loss : P_{loss}
RF loss at Input port : P_e
RF loss at pickup port : P_t



nergy conservation law :

$$P_{loss} = P_{in} - P_t - P_r$$



Q_0 is contributed from only **Cavity Wall Loss**
Q_L is contributed "Cavity Wall Loss" and power loss leaked through **Cables**

o measure Q_O, need to know the ratio (coupling) between wall loss and leakage energy.

Coupling looking from Input port to total system(includes only Pickup Port loss) : β_{in}^*

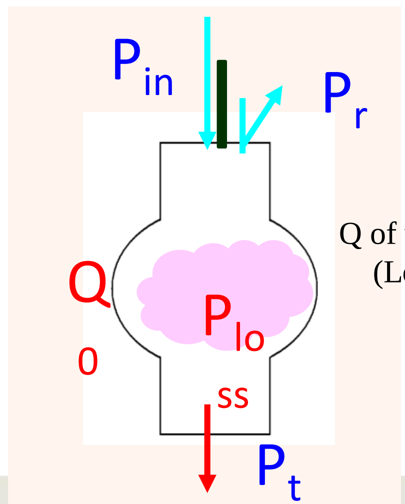
$$\beta_{in}^* = \frac{P_e}{P_{loss}} = \frac{1 \pm \sqrt{\frac{P_r}{P_{im}}}}{1 \mp \sqrt{\frac{P_r}{P_{im}}}} = \frac{1 \pm S_{11}}{1 \mp S_{11}}$$

Over Coupling : $\beta_{in}^* > 1$
Under Coupling: $\beta_{in}^* < 1$

$$P_e = |S_{11}|^2 P_{in} = \left(\frac{1 - \beta_{in}^*}{1 + \beta_{in}^*}\right)^2$$

Coupling of Pickup Port : β_t

$$\beta_t = \frac{P_t}{P_{loss}}$$



Q of whole cavity
(Loaded Q_L)

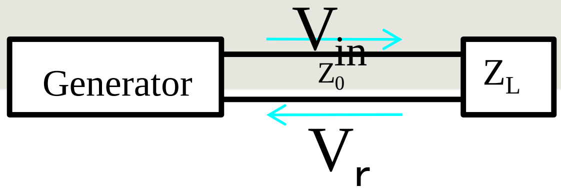
Q_O(Unloaded Q)

$$Q_0 = \left(1 + (1 + \beta_t)\beta^*{}_{in} + \beta_t\right) Q_L$$

Coupling

Thought of equivalent circuit

Example Reflection ratio: Put an article of impedance Z to the cable with Impedance Z_0

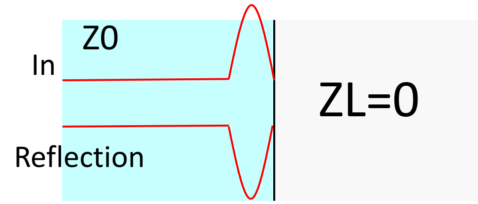


$$\Gamma \equiv \frac{V_r}{V_{in}} \equiv S_{11} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Case of Z_L is shortened

$$Z_L = 0 \quad \text{Reflection ratio} \quad \Gamma \equiv \frac{V_r}{V_{in}} \equiv S_{11} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1$$

Short

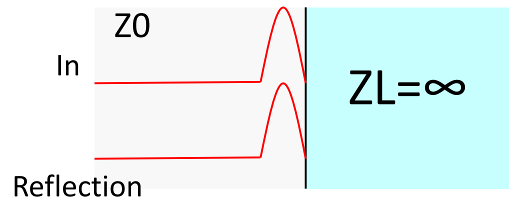


Phase shifts by π

Case of Z_L is Opened

$$Z_L = \infty \qquad \Gamma \equiv \frac{V_r}{V_{in}} \equiv S_{11} = \frac{Z_L - Z_0}{Z_L + Z_0} \approx \frac{\infty}{\infty} = 1$$

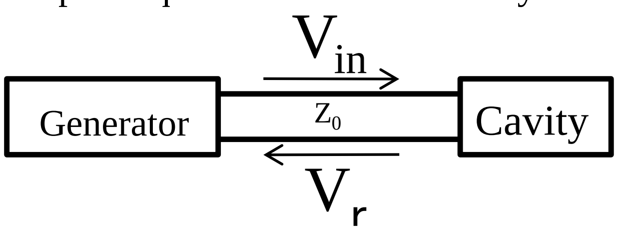
Open

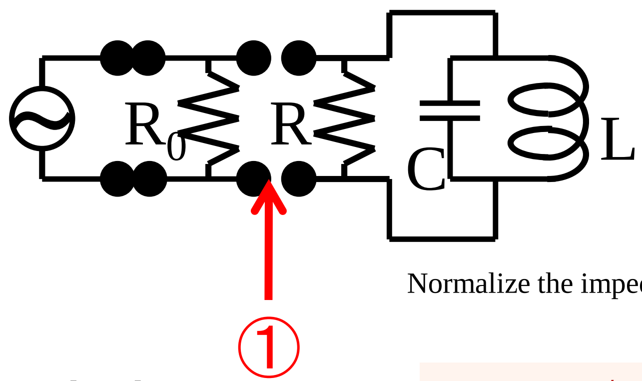


Phase does not shift

Now cavity and transformer

Example Equivalent circuit of cavity and cable





Thought of Smith chart

Normalize the impedance of Cable as Z0=R0=1

Notations of reentrance, condenser, and coil

$$Z_R = R \qquad Z_C = \frac{1}{i\omega C} \qquad Z_C = i\omega L$$

Impedance in parallel connection of hesitance, condenser, and coil.

$$\frac{1}{Z_L} = \frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L} = \frac{1}{R} + \frac{1}{1/i\omega C} + \frac{1}{i\omega L} = \frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right) = X + iY$$

1/ZL : Admittance
X : conductance
Y : suceptance

Digression

Coupling continued

$$\Gamma \equiv \frac{V_r}{V_{in}} \equiv S_{11} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L - 1}{Z_L + 1} = \frac{1 - 1/Z_L}{1 + 1/Z_L} = \frac{1-(X+iY)}{1+(X+iY)} = \frac{1-X-iY}{1+X+iY}$$

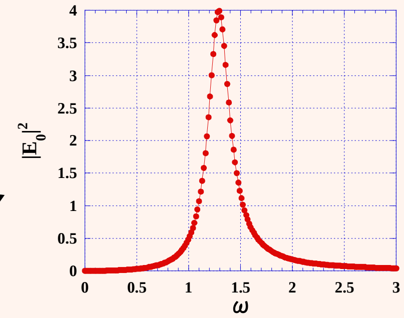
Take real part of the Impedance Z_L

$$\mathrm{Re}\left(\frac{1}{Z_L}\right) = \sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

Voltage excited in the cavity E_0

$$\Rightarrow \quad E_0 = \frac{I_0}{Z_L} = \frac{I_0}{\sqrt{(1/R)^2 + (\omega C - 1/\omega L)^2}} \qquad |E_0|^2 = \frac{I_0^{\ 2}}{(1/R)^2 + (\omega C - 1/\omega L)^2}$$

Lorentz form



Consider Y

$$Y = \omega C - \frac{1}{\omega L}$$

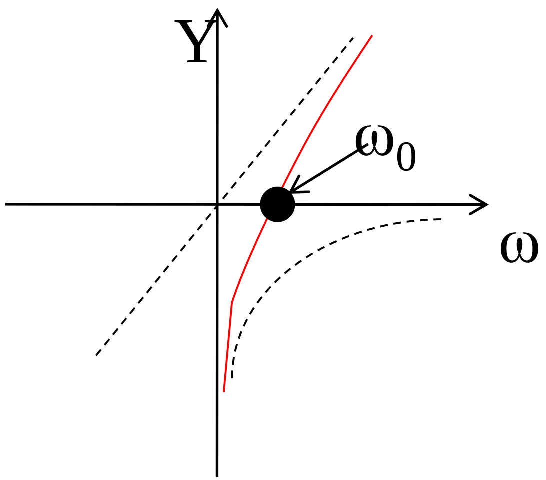
The point far from resonant frequency

$$\omega \to \infty \Rightarrow Y = \infty \Rightarrow S_{11} = \frac{1-X-iY}{1+X+iY} \to -1$$

$$\omega \to 0 \Rightarrow Y = -\infty \Rightarrow S_{11} = \frac{1-X-iY}{1+X+iY} \to -1$$

On resonance

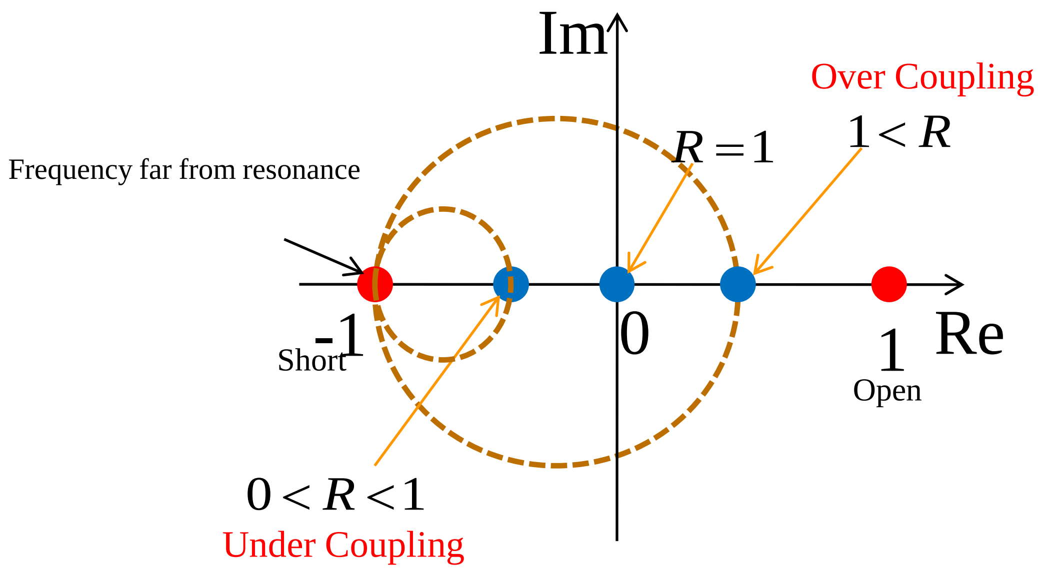
$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow Y = 0 \Rightarrow S_{11} = \frac{1-X-iY}{1+X+iY} \to \frac{1-X}{1+X}$$

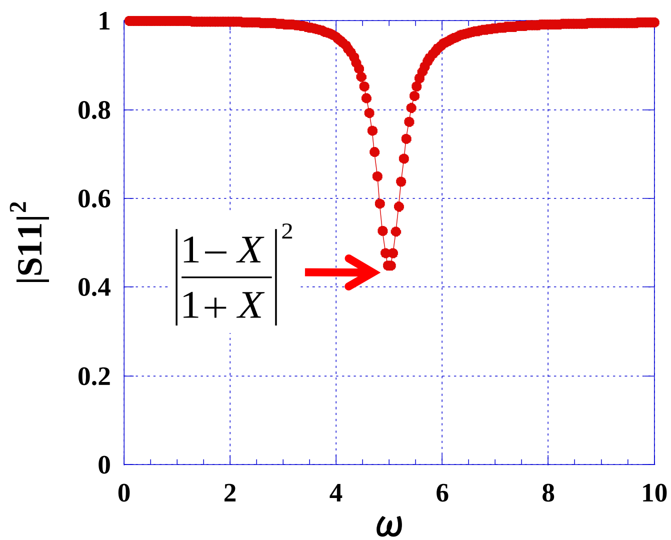


Coupling continued

Consider reflection complex plane

$$|S_{11}|^2 = \frac{(1-X)^2 + (\omega C - 1/\omega L)^2}{(1+X)^2 + (\omega C - 1/\omega L)^2}$$





Cavity resonant frequency ω0

$$S_{11}(\omega_0) = \frac{1 - X - iY}{1 + X + iY} \rightarrow \frac{1-X}{1+X} = \frac{R-1}{R+1}$$

Only real part

$0 < R < 1$ $\quad -1 < S_{11}(\omega_0) < 0 \longrightarrow$ Under

$R = 1$ $\quad S_{11}(\omega_0) = 0$

$1 < R$ $\quad 0 < S_{11}(\omega_0) < 1 \longrightarrow$ Over

R is normalized by R0. R0 is the Cable impedance. Writing R0 explicitly

$$S_{11}(\omega_0) = \frac{R/R_0 - 1}{R/R_0 + 1} = \frac{\beta - 1}{\beta + 1}$$

β is called "coupling".

Coupling continued

S_{11} moves on the circle of the complex plane

$$S_{11}=\frac{1-X-iY}{1+X+iY}=\frac{(1-X-iY)(1+X-iY)}{(1+X)^2+Y^2}=\frac{(1-X)(1+X)-Y^2-iY(1+X+1-X)}{(1+X)^2+Y^2}$$

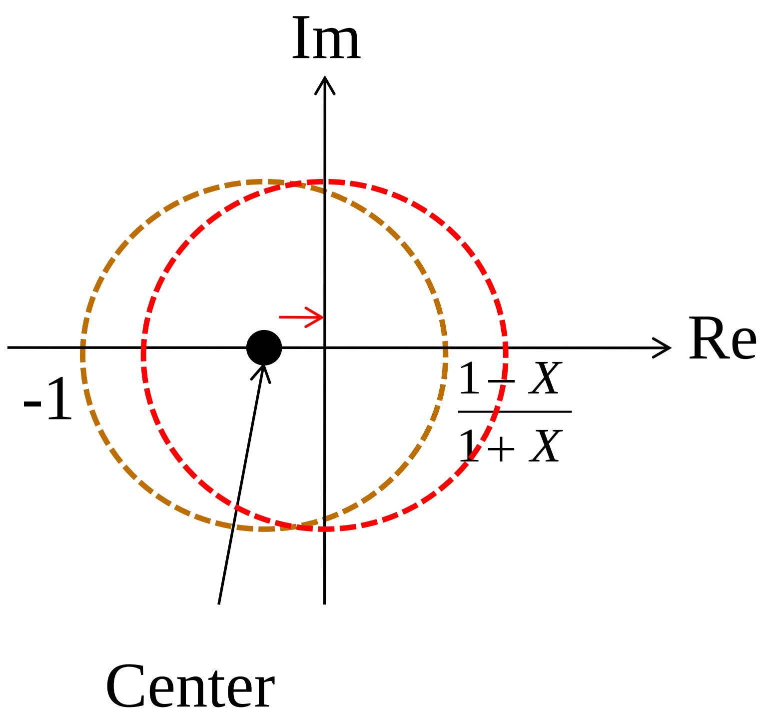
$$=\frac{1-X^2-Y^2-i\,2Y}{(1+X)^2+Y^2}$$

Diameter of the circle

$$\frac{1-X}{1+X}-(-1)=\frac{1-X+1+X}{1+X}=\boxed{\frac{2}{1+X}}$$

Center of the circle

$$-1+\frac{1}{1+X}=\frac{-1-X+1}{1+X}=\boxed{-\frac{X}{1+X}}$$



The center of circle center of S11 locate on the Re axis

$$\mathrm{Re}(S_{11})-\left(-\frac{X}{1+X}\right)=\frac{1-X^2-Y^2}{(1+X)^2+Y^2}+\frac{X}{1+X}$$

$$=\frac{(1-X^2-Y^2)(1+X)+X\left\{(1+X)^2+Y^2\right\}}{\left\{(1+X)^2+Y^2\right\}(1+X)}$$

$$=\frac{(1-X^2-Y^2+X-X^3-XY^2)+(X+2X^2+X^3+XY^2)}{\left\{(1+X)^2+Y^2\right\}(1+X)}$$

Coupling continued

$$\mathrm{Re}(S_{11})-\left(-\frac{X}{1+X}\right)=\frac{1+X^2+2X-Y^2}{\left\{(1+X)^2+Y^2\right\}(1+X)}=\frac{(1+X)^2-Y^2}{\left\{(1+X)^2+Y^2\right\}(1+X)}$$

$$\mathrm{Im}(S_{11})=\frac{-2Y}{(1+X)^2+Y^2}$$

If this draws circle, we can obtain the following relationship

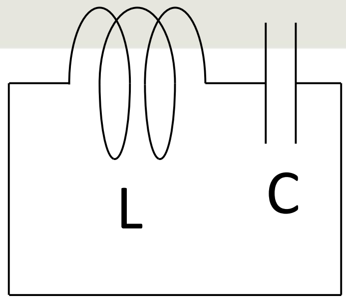
$$\left[\mathrm{Re}(S_{11})-\left(-\frac{X}{1+X}\right)\right]^2+\left[\mathrm{Im}(S_{11})\right]^2=\left(\frac{1}{1+X}\right)^2$$

$$\left[\frac{(1+X)^2-Y^2}{\left\{(1+X)^2+Y^2\right\}(1+X)}\right]^2+\left[\frac{-2Y}{(1+X)^2+Y^2}\right]^2=\left(\frac{1}{1+X}\right)^2$$

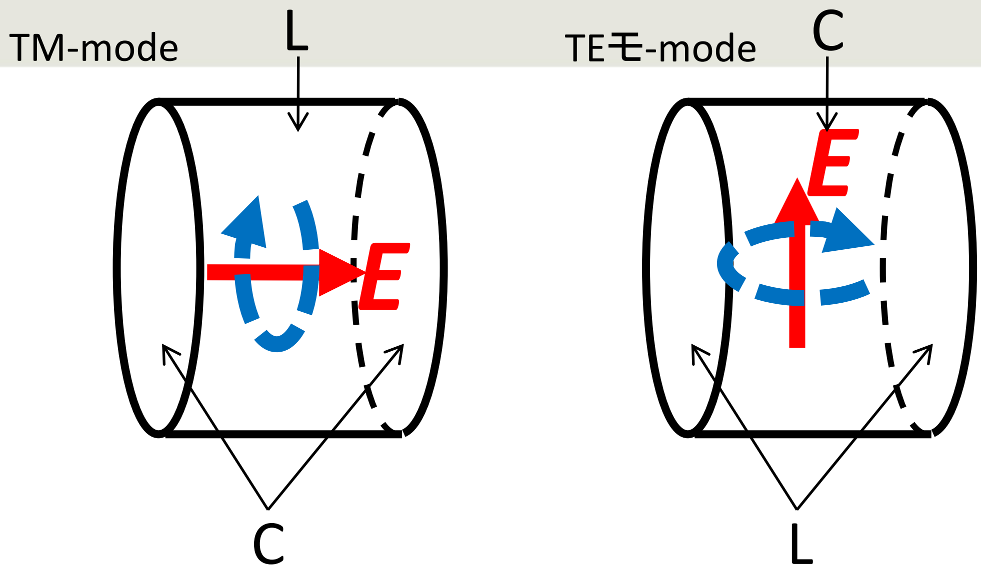
$$\Rightarrow\ \frac{\left\{(1+X)^2-Y^2\right\}^2+4Y^2(1+X)^2}{\left\{(1+X)^2+Y^2\right\}^2(1+X)^2}=\frac{\left\{(1+X)^2+Y^2\right\}^2}{\left\{(1+X)^2+Y^2\right\}^2(1+X)^2}=\frac{1}{(1+X)^2}$$

S11 moves on the circle indepedent on Y

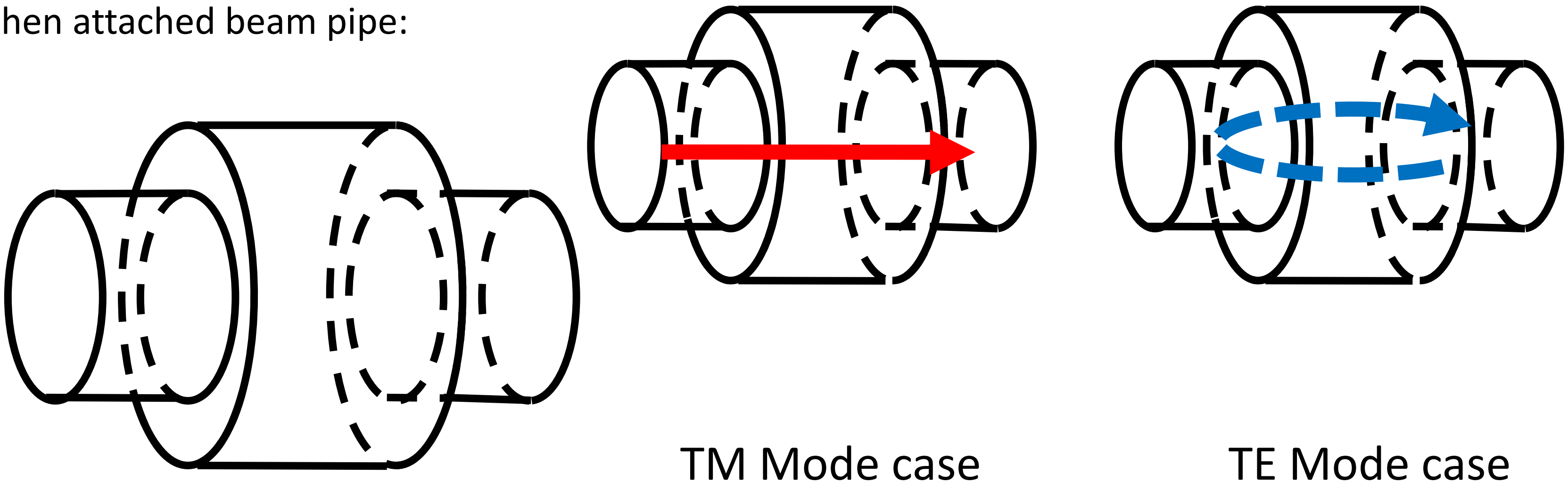
Equivalent circuit for Pillbox cavity with beam pipe



$$f = \frac{1}{2\pi\sqrt{LC}}$$



When attached beam pipe:

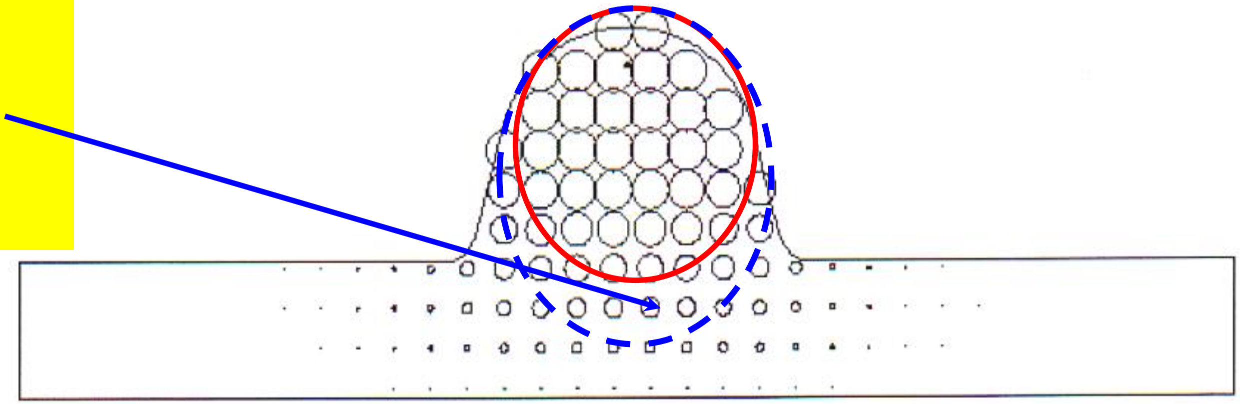


TM Mode case
C reduces⇒ f increase

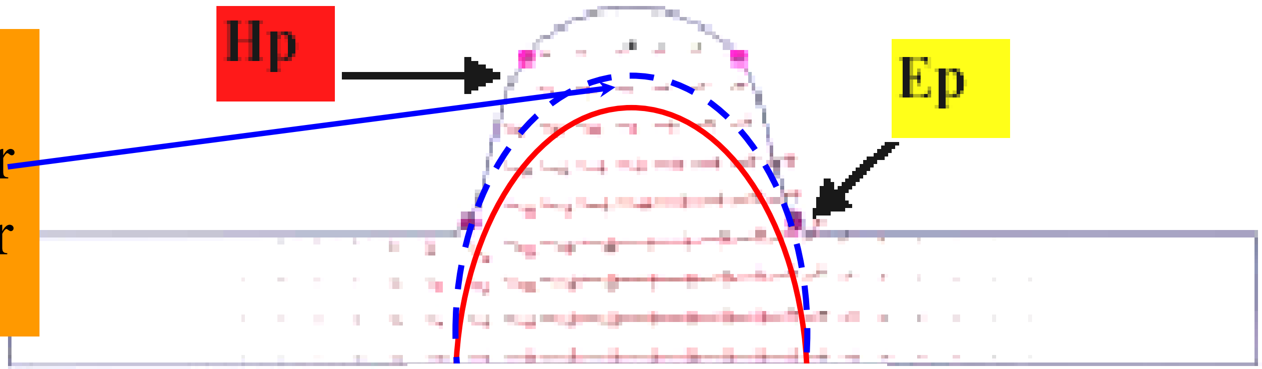
TE Mode case
L increases ⇒ f decreases

Intuitive understanding RF characteristics of cavity

BP ⟶ larger
Area of H ⟶ Larger
Hp/Eacc ⟶ larger

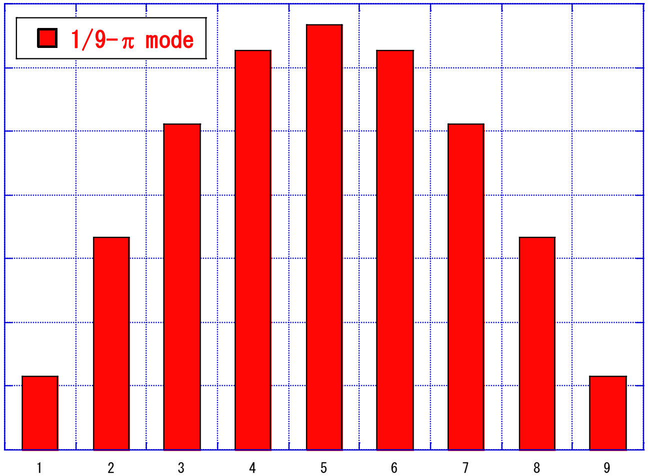


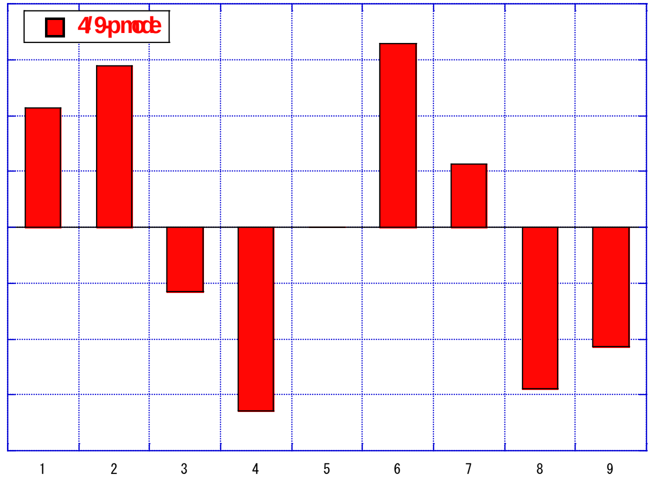
BP ⟶ **smaller**
Area of E ⟶ Larger
Ep/Eacc ⟶ smaller

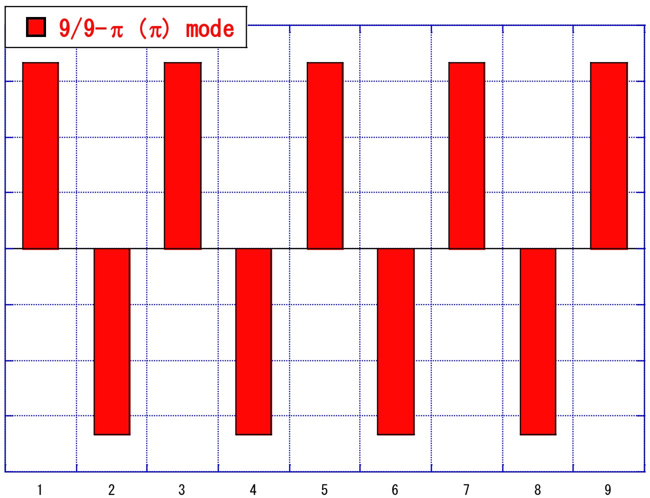


The diameter of beam pipe impacts on cavity RF parameters. The smaller the sizes, the Hp/Eacc, Ep/Eacc, become smaller. In addition, when E-field concentrates more on beam axis and the shunt impedanc increases.
However, in the multi cell structure the cell-to-cell coupling becomes small and field becomes unstable. HOM is also problematic. Consider these behavior, the multi-cell structure is optimized.

Field distribution in pass-bands



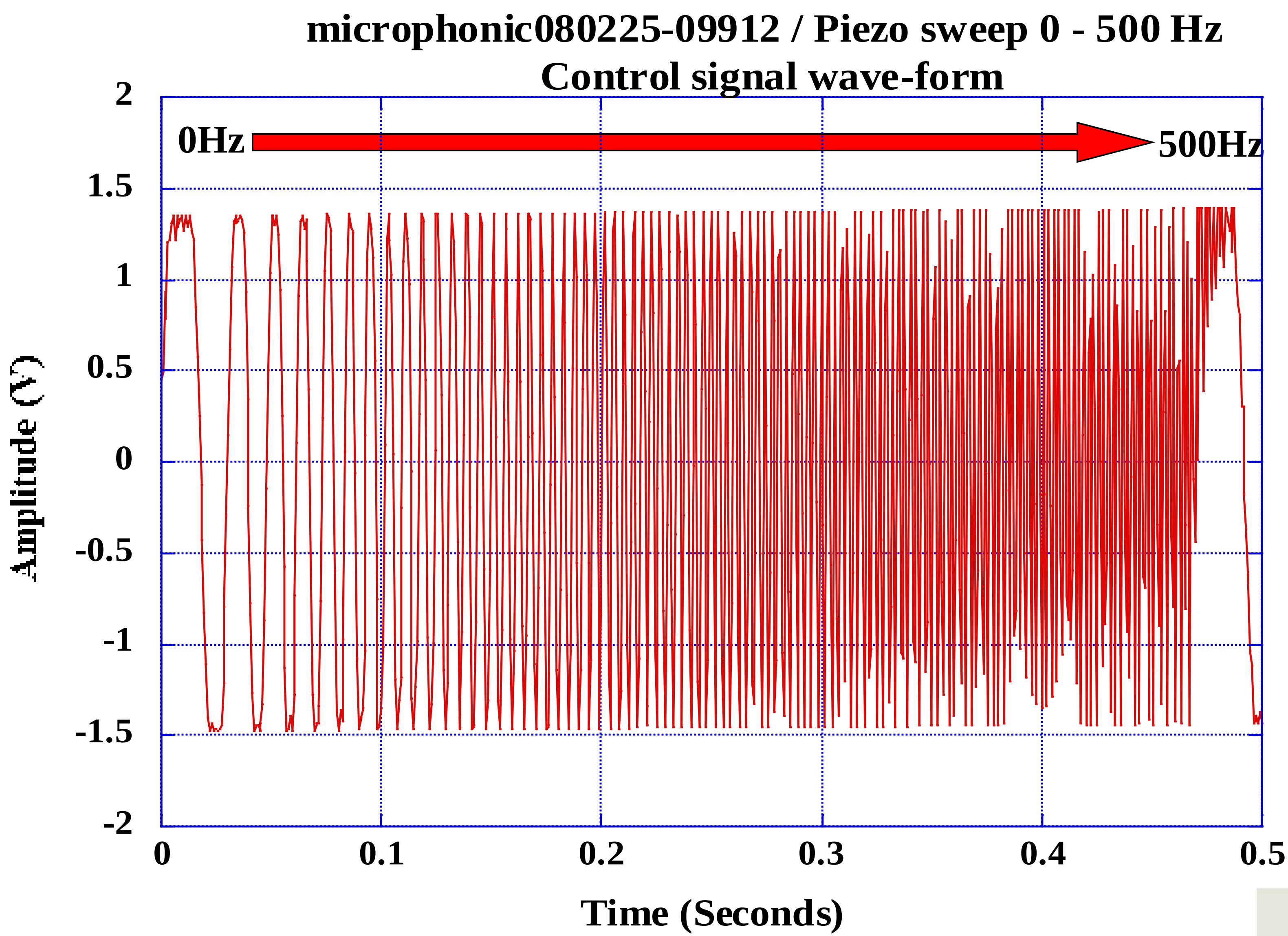




Phase advance by mπ/N per one cavity (N cell).

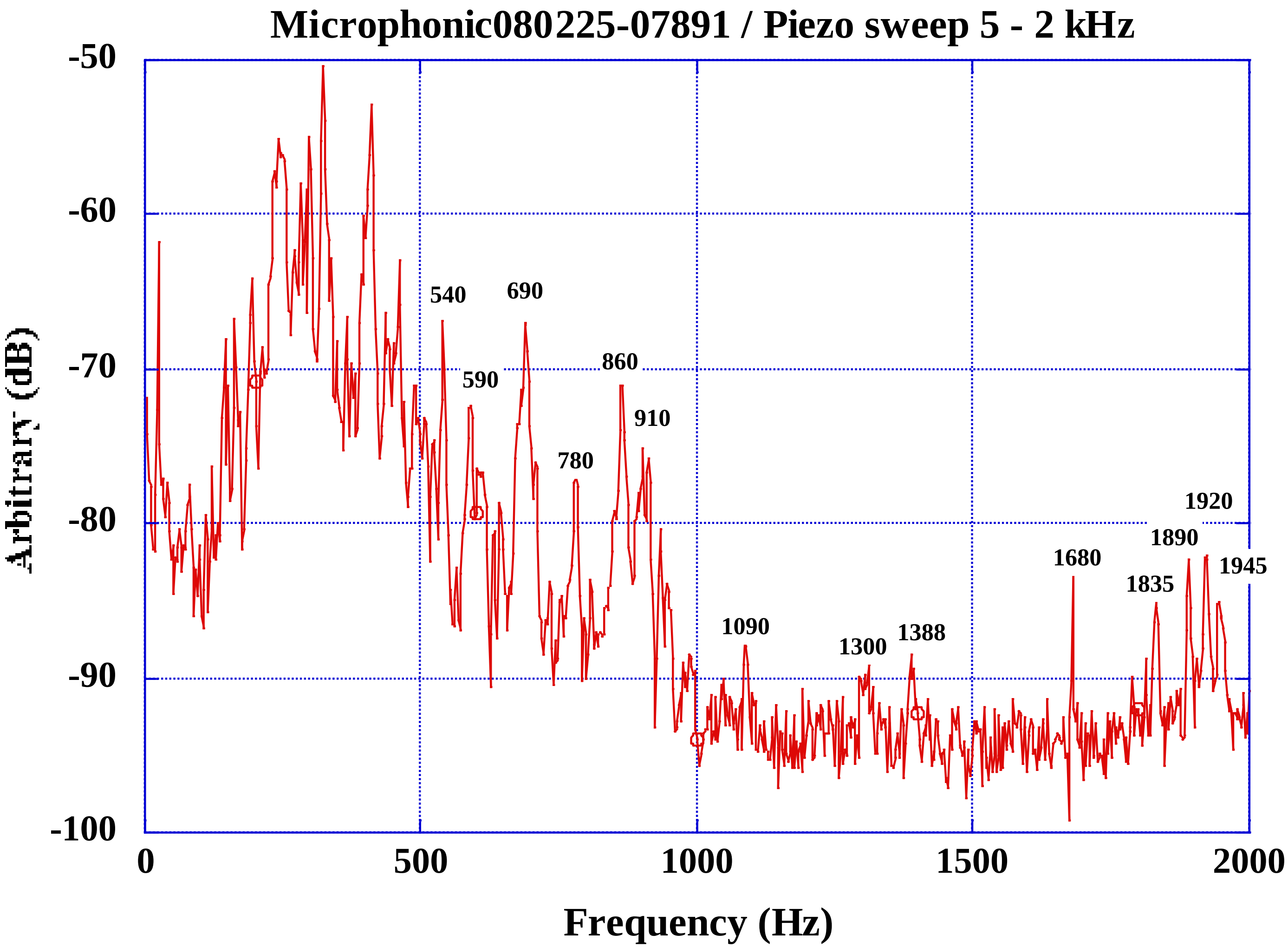
Piezo Excitation by f-sweep mode (Example)

Example of Piezo excitation



Mechanical Resonance Scan (0 – 2kHz)

FFT signal



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